Black Holes and their Thermodynamics[1]

Problem Sheet 12

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Tutorials take place on Mondays, 2-4pm (c.t.) in room A 449, Theresienstr. 37. Please hand in your solutions at the next tutorial on January 27, 2020 or send them in a single pdf in an email before the tutorial.

Exercise 1 *One goal of string theory in* 10 *dimensions is to describe lower-dimensional effective physics. In this exercise we look at* 5*-dimensional black hole spacetimes coming from a type IIB compactification.*

In order to describe 5*-dimensional effective physics, one needs to get rid of the remaining* $10-5$ *so-called internal dimensions. The idea is to compactify the internal dimensions and to describe their dynamics by fields living on the* 5*-dimensional subspace. E.g. if one compactifies the* 10*-dimensional spacetime to* $\mathbf{R}^{4,1}\times T^5$*, where* $T^5 = S^1\times ...\times S^1$ *is the* 5*-dimensional Torus, then one may describe the size of this internal spacetime by radii-fields corresponding to the individual circles* S^1 *that live on* $\mathbf{R}^{4,1}$*. (The radii are sometimes referred to as moduli of the internal space.) Additional effective physics may be obtained by turning on fluxes or placing objects such as branes in the internal dimensions.*

One way to construct an effective 5*-dimensional black hole, is to wrap a configuration of branes in a specific way around the internal space* T^5*. E.g. a black hole with three charges in five dimensions can be obtained by taking* Q_1 *D*1*-branes wrapped on an* S^1 *of radius* R *inside the* T^5*,* Q_5 *D*5*-branes wrapped on the* $T^5 = T^4\times S^1$*, and* n *units of Kaluza-Klein momentum along the same circle* S^1*.*

In Einstein form, the resulting 5-dimensional metric takes the form

$$\mathrm{d}s^2 = -\lambda^{-2/3}\mathrm{d}t^2 + \lambda^{1/3}(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_3^2) \tag{1}$$

where

$$\lambda = \prod_{i=1}^{3}\left[1+\left(\frac{r_i}{r}\right)^2\right]. \tag{2}$$

The horizon of the black hole is located at $r=0$*. Note that this metric reduces to the* 5*D-extremal Reissner-Nordström black hole when* $r_1=r_2=r_3$*. The*

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r_i are related to the masses M_i of the branes through

$$r_i^2 = \frac{g_s^2 l_s^8}{RV} M_i \tag{3}$$

where $(2\pi)^4 V$ is the volume of the T^4 and R is the radius of S^1. And by string theoretic considerations, the masses can be determined to be

$$M_1 = \frac{Q_1 R}{g_s l_s^2}\,,\ M_2 = \frac{Q_5 RV}{g_s l_s^6}\,,\ M_3 = \frac{n}{R}\,. \tag{4}$$

1. *Compute the area of the horizon of this black hole and express the result in terms of Q_1, Q_5 and n. Think of a reason why it is necessary to wrap three different objects around the internal dimensions in order to construct a black hole.*

Counting the degeneracy (microstates) of the black hole state means counting all possible configurations of strings attached to a given set of intersecting branes that leave the macroscopic value of energy, mass and charges invariant, i.e. one has to count the number of states of the open strings stretching between the $D1$-branes and the $D5$-branes that carry the momentum n/R along the compact direction common to all the branes.

2. *Follow the reasoning below Equation (155) of https://arxiv.org/pdf/hep-th/9904033.pdf to argue that the entropy obtained through microstate counting of this brane configuration is one quarter of the area of the horizon.*

The metric of a $5D$-supersymmetric rotating black hole is a relatively simple generalization of the above metric, given by

$$\mathrm{d}s^2 = \lambda^{-2/3}\left(\mathrm{d}t - \frac{a}{r^2}\sin^2\theta\mathrm{d}\phi + \frac{a}{r^2}\cos^2\theta\mathrm{d}\psi\right)^2 + \lambda^{1/3}\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_3^2\right) \tag{5}$$

where λ is given above. In Euclidean coordinates,

$$\begin{aligned} x_1 &= r\cos\theta\cos\psi \\ x_2 &= r\cos\theta\sin\psi \\ x_3 &= r\sin\theta\cos\phi \\ x_4 &= r\sin\theta\sin\phi \end{aligned}$$

this metric describes simultaneous rotation along the 12 and 34 planes. The parameter a is related to $J_{12} = J_{34} = J$ by

$$J = \frac{\pi a}{4G_5} \tag{6}$$

with the 5-dimensional Newton constant G_5.

3. *By taking the near horizon limit of this metric, compute the area of the horizon of this rotating black hole and deduce its entropy.*

4. *Does string theory provide a microscopical explanation for the entropy of this black hole? Does string theory provide an explanation for the entropy of rotating black holes in four dimensions? (You may want to consult page 573 and 574 of the book "String theory and M-theory: A modern introduction" by Becker, Becker and Schwarz.)*