Black Holes and their Thermodynamics[1]

Problem Sheet 11

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Tutorials take place on Mondays, 2-4pm (c.t.) in room A 449, Theresienstr. 37. There will be no tutorials on January 13 and 20. Please hand in your solutions at the next tutorial on January 27, 2020 or send them in a single pdf in an email before the tutorial.

Exercise 1 *Consider a massless scalar field in $1+1$-dimensional Minkowski spacetime. The standard mode expansion for the field operator in light-cone coordinates (u, v) is given by*

$$\hat{\phi} = \int_0^\infty \frac{\mathrm{d}\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \left(\hat{a}_\omega^- e^{-i\omega u} + \hat{a}_\omega^+ e^{i\omega u} \right) + (\text{left moving}) \tag{1}$$

with the creation and annihilation operators $\hat{a}_\omega^+$ and $\hat{a}_\omega^-$, resp. Starting from Rindler coordinates, and defining light-cone coordinates $(\tilde{u}, \tilde{v})$ from there, the standard mode expansion can also be written

$$\hat{\phi} = \int_0^\infty \frac{\mathrm{d}\Omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\Omega}} \left(\hat{b}_\Omega^- e^{-i\Omega\tilde{u}} + \hat{b}_\Omega^+ e^{i\Omega\tilde{u}} \right) + (\text{left moving}) \tag{2}$$

with the creation and annihilation operators $\hat{b}_\Omega^+$ and $\hat{b}_\Omega^-$, resp. The operators enjoy the following commutation relations

$$[\hat{a}_\omega^-, \hat{a}_{\omega'}^+] = \delta(\omega - \omega')\ , \quad [\hat{b}_\Omega^-, \hat{b}_{\Omega'}^+] = \delta(\Omega - \Omega')\,. \tag{3}$$

1. *Show that the creation and annihilation operators $\hat{a}_\omega^\pm$ and $b_{\Omega'}^\pm$ are related by a Bogoliubov transformation of the form*

$$\hat{b}_\Omega^- = \int_0^\infty \mathrm{d}\omega \left(\alpha_{\Omega\omega} \hat{a}_\omega^- - \beta_{\Omega\omega} \hat{a}_\omega^+ \right) \tag{4}$$

such that

$$\int_0^\infty \mathrm{d}\omega \left(\alpha_{\Omega\omega} \alpha_{\Omega'\omega}^* - \beta_{\Omega\omega} \beta_{\Omega'\omega}^* \right) = \delta(\Omega - \Omega') \tag{5}$$

Derive expressions for the matrices $\alpha_{\Omega\omega}$ and $\beta_{\Omega\omega}$. Comment on invertibility of this transformation.

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[1] *www.physik.uni−muenchen.de/lehre/vorlesungen/wise_19_20/bh_info_wise_2019_20*

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2. *Using your previous result, show that*

$$|\alpha_{\Omega\omega}|^2 = e^{\frac{2\pi\Omega}{a}} |\beta_{\Omega\omega}|^2 \tag{6}$$

3. *Compute the expectation value of the b-particle number operator $\hat{N}_\Omega = \hat{b}^+_\Omega \hat{b}^-_\Omega$ in the Minkowski vacuum $|0_M\rangle$ defined by $\hat{a}^-_\omega |0_M\rangle = 0$. Verify that the particle density n_Ω is given by*

$$n_\Omega = \frac{1}{e^{\frac{2\pi\Omega}{a}} - 1} \tag{7}$$

and read off the temperature associated with this Bose-Einstein distribution. What is a?

4. *Compare the mode expansion of a scalar field in Kruskal and Eddington-Finkelstein coordinates and use the previous results to show that Hawking radiation is analogoues to the Unruh effect with $a = \kappa$.*

Exercise 2 *Make a proposal for a Penrose diagram of an evaporating Schwarzschild black hole. Use it to explain the information loss paradox.*

Exercise 3 *The classical area theorem due to Hawking (1972) states that the surface area A of a black hole does not decrease in time, $\mathrm{d}A/\mathrm{dt} \geq 0$. However, Hawking radiation reduces the area of a black hole violating the second law of black hole mechanics.*

Hawking's area theorem holds assuming the null energy condition (NEC). It states that null observers only see non-negative energy densities

$$T_{kk} = T_{ab} k^a k^b \geq 0 \ \ \forall\ k^2 = 0\,, \tag{8}$$

where T_{ab} is the energy-momentum tensor and k^a is a null vector. However, quantum states, such as Hawking radiation, violate the NEC.

In this exercise we will familiarize ourselves with a possible resolution. The starting point is to define a generalized or quantum corrected entropy

$$S_{\text{gen}} = S_{\text{BH}} + S_{\text{out}}\,, \tag{9}$$

where $S_{\text{BH}} = \frac{A}{4G\hbar}$ is the classical black hole entropy. The other term is the entropy of the region outside the black hole and given by the von Neumann entropy $S_{\text{out}} = -\mathrm{Tr}\rho \ln \rho$ of the quantum state ρ. On this basis, already Bekenstein proposed the Generalized Second Law (GSL)

$$\mathrm{d}S_{\text{gen}} \geq 0\,, \tag{10}$$

in order to incorporate the entropy of matter that falls into the black hole and not violate the Second Law of Thermodynamics outside the black hole.

The first step is to define a quantum expansion Θ*. Let us first introduce the expansion* θ *in classical general relativity. Take a congruence of null geodesics. Let* $\mathcal{A}$ *be an area element orthogonal to the congruence. Then the expansion is given by*

$$\theta = \frac{\mathcal{A}'}{\mathcal{A}} \tag{11}$$

where $\mathcal{A}'$ *is the derivative of the area w.r.t. the affine parameter of the the null congruence. I.e.* θ *measures the change of the area along the null congruence normalized to the area.*

To define the quantum expansion take a hypersurface Σ*, e.g. the surface of a black hole, and deform it by an infinitesimal amount* $\mathcal{A}$ *along a orthogonal direction. Then we define*

$$\Theta = \lim_{\mathcal{A}\to 0} \frac{4G\hbar}{\mathcal{A}} S'_{\text{gen}} \tag{12}$$

where S'_{gen} *denotes derivative w.r.t. the localized deformation of the surface (we will use this highly simplified notation in the following. It really represents directional functional derivatives.).*

Classically, the NEC implies that $\theta' \leq 0$ *(Why?), which is referred to as focussing theorem. It means that light rays can only be focussed and never anti-focussed if the NEC is satisfied. This motivates the following Quantum Focussing Conjecture (QFC)*

$$\Theta' \leq 0\,. \tag{13}$$

The QFC is the basis for several interesting results in quantum gravity.

1. *Convince yourself that the quantum expansion can be written as*

$$\Theta = \theta + \lim_{\mathcal{A}\to 0} \frac{4G\hbar}{\mathcal{A}} S'_{\text{out}}\,. \tag{14}$$

2. *Show that the derivative of the quantum expansion w.r.t. the localized deformation of the surface is given by*

$$\Theta' = \theta' + \lim_{\mathcal{A}\to 0} \frac{4G\hbar}{\mathcal{A}} \left(S''_{\text{out}} - S'_{\text{out}}\theta\right)\,. \tag{15}$$

3. *By using Raychauduri's equation and choosing an appropriate null congruence with tangent vector k^a show that the QFC implies*

$$\langle T_{kk} \rangle \geq \lim_{\mathcal{A} \to 0} \frac{\hbar}{2\pi \mathcal{A}} S''_{\text{out}} \,. \tag{16}$$

This inequality is referred to as quantum null energy condition (QNEC).

A large class of (bosonic) QFTs, free and interacting, are proven to satisfy the QNEC. However, these proofs are non-trivial. Note that the dependence on Newton's constant G dropped out in this expression.

4. *Now take a black hole. Give an argument on why one can expect that $\Theta \to 0$ in the asymptotic future. With this assumption, use the QFC to deduce the GSL.*

Hints: *It might be instructive to read the pedagogical paper arXiv:1810.01880 and go through parts of the technical paper arXiv:1506.02669.*