Black Holes and their Thermodynamics[1]

Problem Sheet 10

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Tutorials take place on Mondays, 2-4pm (c.t.) in room A 449, Theresienstr. 37. There will be no tutorial on December 23. Please hand in your solutions at the next tutorial on January 10, 2020 or send them in a single pdf in an email before the tutorial. Note the unusual date and time.

Exercise 1 *The Kerr-Newman black hole with electrical charge Q is given by*

$$\mathrm{d}s^2 = -\frac{\rho^2\Delta}{\Sigma}\mathrm{d}t^2 + \frac{\Sigma}{\rho^2}\sin^2\theta\,(\mathrm{d}\phi - \omega\mathrm{d}t)^2 + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 \tag{1}$$

$$A_t = \frac{Qr}{\rho^2}\,,\ A_\phi = -\frac{Qar\sin^2\theta}{\rho^2} \tag{2}$$

where

$$\Delta = r^2 - 2Mr + a^2 + Q^2\,,\ \rho^2 = r^2 + a^2\cos^2\theta\,, \tag{3}$$

$$\Sigma = (r^2+a^2)^2 - a^2\Delta\sin^2\theta\,,\ \omega = \frac{(2Mr - Q^2)a}{\Sigma} \tag{4}$$

1. *The co-rotating electric horizon potential is defined by*

$$\Phi_H = \xi^\mu A_\mu\Big|_{r=r_+}\,,\text{with } \xi = \partial_\mathrm{t} + \Omega_\mathrm{H}\partial_\phi\,, \tag{5}$$

where $\Omega_H = \frac{a}{r_+^2+a^2}$. Here we assume that $M^2 > a^2 + Q^2$ and r_+ is the radial position of the outer horizon of the Kerr-Newman black hole. Determine r_+. Show that

$$\Phi_H = \frac{Qr_+}{r_+^2 + a^2} \tag{6}$$

2. *Convince yourself that the area of the (outer) event horizon is still given by $A = 4\pi(r_+^2 + a^2)$ and verify Smarr's formula*

$$M = \frac{\kappa A}{4\pi} + 2\Omega_H J + \Phi_H Q\,, \tag{7}$$

where $\kappa = \frac{r_+ - r_-}{2(r_+^2+a^2)}$ is the surface gravity of the outer event horizon.

[1] $www.physik.uni-muenchen.de/lehre/vorlesungen/wise\_19\_20/bh\_info\_wise\_2019\_20$

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3. *Use this formula for M to deduce the first law of black hole mechanics for charged rotating black holes:*

$$\mathrm{d}M = \kappa \mathrm{d}A + \Omega_H \mathrm{d}J + \Phi_H \mathrm{d}Q\,. \tag{8}$$

Exercise 2 *In this exercise we will study implications of Hawking's area theorem.*

1. *The area law implies a limit to the efficiency of mass/energy conversion in black hole collisions. Consider two black holes that initially are well separated so that we can approximate them as Schwarzschild black holes (i.e. uncharged and non-rotating) of masses M_1 and M_2, respectively. Assume further that the final black hole is again a Schwarzschild black hole of mass M_3. The energy difference $E = M_1 + M_2 - M_3$ is radiated away by gravitational radiation. Use the black hole area law to show*

$$\eta \equiv \frac{E}{M_1 + M_2} \leq 1 - \frac{1}{\sqrt{2}} \tag{9}$$

   for the efficiency of the process.

2. *Assuming the area law show that a Schwarzschild black hole of mass M_3 can never bifurcate into two Schwarzschild black holes of masses M_1 and M_2, respectively.*

Exercise 3 *Identifying the surface gravity κ of the (outer) event horizon with the temperature of a black hole via*

$$T = \frac{\hbar}{2\pi}\kappa \tag{10}$$

leads to a negative heat capacity for a Schwarzschild black hole (Check this!), i.e. the temperature decreases with increasing mass. Plot $T/\hbar$ as a function of M/Q for a Reissner-Nordström black hole for a fixed charge Q, and discuss the heat capacity in that case.

*(**Hint:** The heat capacity of a charged black hole of mass M and fixed charge Q is $C = T\frac{\partial S}{\partial T}|_Q$ where $S = A/4\hbar$ is the Bekenstein-Hawking entropy of the black hole.)*

Exercise 4 *Consider the Klein-Gordon equation of a massive, complex scalar field in Minkowski spacetime,*

$$(\nabla^2 - m^2)\phi = 0\,. \tag{11}$$

1. *Show that the inner product on the space of solutions to the Klein-Gordon equation, i.e.*

$$\langle \phi_1, \phi_2 \rangle = -i \int_{\Sigma_t} \mathrm{d}^{D-1}x (\phi_1 \partial_t \phi_2^* - \phi_2^* \partial_t \phi_1) \tag{12}$$

is independent of the constant-time hypersurface Σ_t *over which the integral is taken.*

2. *Use the inner product (11) to show*

$$\langle f_{\vec{k}_1}, f_{\vec{k}_2} \rangle = -\langle f^*_{\vec{k}_1}, f^*_{\vec{k}_2} \rangle = \delta^{D-1}(\vec{k}_1 - \vec{k}_2)\,,\ \langle f_{\vec{k}_1}, f^*_{\vec{k}_2} \rangle = 0\,, \tag{13}$$

with

$$f_{\vec{k}} = \frac{e^{ik_\mu x^\mu}}{\sqrt{(2\pi)^{D-1} 2\omega}}\,,\ \omega^2 = \vec{k}^2 + m^2\,,\ \omega > 0 \tag{14}$$

3. *Argue that the vacuum state does not change under a Lorentz transformation.*