**Black Holes and their Thermodynamics**[1]

# Problem Sheet 1

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Tutorials take place on Mondays, 2-4pm (c.t.) in room A 449, Theresienstr. 37. Please hand in your solutions at the next tutorial on October 21, 2019.

**Exercise 1** *The line element of Minkowski space in inertial coordinates* $(t, x, y, z)$ *is given by*

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \,. \tag{1}$$

1. *Calculate the line element in spherical coordinates* $(t, r, \theta, \phi)$ *defined by*

$$r^2 = x^2 + y^2 + z^2 \,, \quad \cos\theta = \frac{z}{r} \,, \quad \tan\phi = \frac{y}{x} \,. \tag{2}$$

   *Also compute the components of the inverse metric.*

2. *Take the line element*

$$\mathrm{d}s^2 = (a^2t^2 - c^2)\mathrm{d}t^2 + 2at\mathrm{d}t\mathrm{d}x + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \,, \tag{3}$$

   *where* $a$ *and* $c$ *are constants. Calculate the components of the inverse metric. By identify a suitable coordinate transformation, show that the line element can be reduced to the Minkowski line element.*

**Exercise 2** *Consider the* 2*-sphere* $S^2$ *with line element*

$$\mathrm{d}s^2 = r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \,, \tag{4}$$

*where* $r$ *is constant and the coordinates are* $(\theta, \phi)$*.*

1. *Please calculate the Riemann-tensor, Ricci-tensor, and Ricci-scalar. Also calculate the Einstein tensor and give a physical interpretation for your result.*

2. *For a circle of diameter* $d$ *on* $S^2$*, calculate its circumference* $C$*. Calculate the ratio of the circumference* $C$ *and the diameter* $d$*,*

$$c = \frac{C}{d} \,. \tag{5}$$

   *What is* $c$ *in Euclidean space?*

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3. *Extra: Now let $g_{ab}$ be the metric of an arbitrary two-dimensional manifold $\mathcal{M}$. Show that the Riemann tensor can be written as*

$$R_{abcd} = K(g_{ac}g_{bd} - g_{ad}g_{bc}) \tag{6}$$

*for some scalar function $K$. Determine $K$. What are the components of the Einstein tensor? Give a physical interpretation of your result.*

***Hint:*** *Use the symmetries of the Riemann tensor ($R_{abcd} = -R_{abdc} = -R_{bacd}$) to count the number of independent components. Then show, that the tensor given in eq. (6) satisfies all symmetry relations of the Riemann tensor. To find $K$, calculate the Ricci scalar from eq. (6).*

**Exercise 3** *On a generic curved manifold, free particles travel along geodesics. Consider the Lagrangian of the geodesic equation*

$$\mathcal{L} = -g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \tag{7}$$

*for a timelike $\mathrm{d}x^\mu/\mathrm{d}\tau$, where $\tau$ is the proper time along the geodesic. Derive the geodesic equation from the Lagrangian through variation.*

*In the Newtonian limit, the metric can be written as*

$$\mathrm{d}s^2 = -(1+2\Phi)\mathrm{d}t^2 + (1-2\Phi)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\,, \tag{8}$$

*where $x^i$ are the spatial coordinates and $\delta_{ij}$ is the Kronecker symbol and $\Phi = \Phi(x^i)$ depends only on the spatial directions. Assuming small velocities and $|\Phi| \ll 1$, show that the geodesic equation reduces the the Newtonian equation of motion for a particle moving in an external gravitational field.*