Prof. Dr. Viatcheslav Mukhanov Winter term 2019/2020

Exercises on Quantum Mechanics II (TM1/TV)

Problem set 3, discussed November 4 - November 8, 2019

Exercise 19

(i) Show that the momentum eigenstates $|\,p\rangle$ are "normalized to the δ-function" if

$$\langle p\,|\,x\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar}px}$$

(ii) Show

$$\hat{x}\,|\,\psi\rangle = \int dx\ x\psi(x)\,|\,x\rangle = i\hbar\int dp\frac{\partial\psi(p)}{\partial p}\,|\,p\rangle$$

and

$$\hat{p}\,|\,\psi\rangle = \int dp\ p\psi(p)\,|\,p\rangle = -i\hbar\int dx\frac{\partial\psi(x)}{\partial x}\,|\,x\rangle$$

where $\psi(x) = \langle x\,|\,\psi\rangle$ and $\psi(p) = \langle p\,|\,\psi\rangle$.

Solution

(i)

$$\langle p'\,|\,p\rangle = \int dx\,\langle p'\,|\,x\rangle\,\langle x\,|\,p\rangle = \int dx\frac{1}{2\pi\hbar}e^{-\frac{i}{\hbar}x(p'-p)} = \int dy\frac{1}{2\pi}e^{iy(p'-p)} = \delta(p'-p)$$

(ii)

$$\hat{x}\,|\,\psi\rangle = \int dx\ \hat{x}\,|\,x\rangle\,\langle x\,|\,\psi\rangle = \int dx\ x\psi(x)\,|\,x\rangle$$

On the other hand

$$\hat{x}\,|\,\psi\rangle = \int dp\ \hat{x}\,|\,p\rangle\,\langle p\,|\,\psi\rangle = \int dp\int dx\ \hat{x}\,|\,x\rangle\,\langle x\,|\,p\rangle\,\psi(p) = \int dp\int dx\ \frac{x}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}px}\psi(p)\,|\,x\rangle =$$

$$\overset{P.I.}{=} \int dx\frac{x}{\sqrt{2\pi\hbar}}\,|\,x\rangle\left[\frac{\hbar}{ix}e^{\frac{i}{\hbar}px}\psi(p)\Big|_{-\infty}^{\infty} - \int dp\frac{\partial\psi(p)}{\partial p}\frac{\hbar}{ix}e^{\frac{i}{\hbar}px}\right] =$$

$$= i\hbar\int dp\int dx\ \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}px}\,|\,x\rangle\,\frac{\partial\psi(p)}{\partial p} = i\hbar\int dp\frac{\partial\psi(p)}{\partial p}\,|\,p\rangle$$

Similarly for $\hat{p}$:

$$\hat{p}\,|\,\psi\rangle = \int dp\ \hat{p}\,|\,p\rangle\,\langle p\,|\,\psi\rangle = \int dp\ p\psi(p)\,|\,p\rangle$$

Again, on the other hand

$$\hat{p}\,|\,\psi\rangle = \int dx\ \hat{p}\,|\,x\rangle\,\langle x\,|\,\psi\rangle = \int dx\int dp\ \hat{p}\,|\,p\rangle\,\langle p\,|\,x\rangle\,\psi(x) = \int dx\int dp\ \frac{p}{\sqrt{2\pi\hbar}}e^{-\frac{i}{\hbar}px}\psi(x)\,|\,p\rangle =$$

$$\overset{P.I.}{=} \int dp\frac{p}{\sqrt{2\pi\hbar}}\,|\,p\rangle\left[-\frac{\hbar}{ip}e^{-\frac{i}{\hbar}px}\psi(x)\Big|_{-\infty}^{\infty} + \int dx\frac{\partial\psi(x)}{\partial x}\frac{\hbar}{ip}e^{-\frac{i}{\hbar}px}\right] =$$

$$= -i\hbar\int dx\int dp\ \frac{1}{\sqrt{2\pi\hbar}}e^{-\frac{i}{\hbar}px}\,|\,p\rangle\,\frac{\partial\psi(x)}{\partial x} = -i\hbar\int dx\frac{\partial\psi(x)}{\partial x}\,|\,x\rangle$$

Exercise 20 (Central tutorial)

(i) Knowing that $\hat{p}\,|\,p\rangle = p\,|\,p\rangle$ and that $[\hat{q},\hat{p}] = i\hbar$, calculate the matrix element $\langle p_1\,|\,\hat{q}\,|\,p_2\rangle$. You should start by considering the matrix element $\langle p_1\,|\,[\hat{q},\hat{p}]\,|\,p_2\rangle$ and its implications.

Hint: Recall what it has been done in the lecture with $\langle q_1\,|\,\hat{p}\,|\,q_2\rangle$, but this time don't set to zero the constant of integration and keep it.

(ii) How does the previous result change if you replace $\hat{q}$ by $\hat{q} + c(\hat{p})$, where $c(\hat{p})$ is an arbitrary function? What does this mean?

(iii) Consider now a general transformation for the redefinition of position and momentum operator:

$$\hat{Q} = \hat{U}\hat{q}\hat{U}^\dagger \quad , \quad \hat{P} = \hat{U}\hat{p}\hat{U}^\dagger$$

Why is a transformation of the type $\hat{Q} = \hat{U}\hat{q}\hat{V}$ (same for $\hat{P}$) with $\hat{V} \neq \hat{U}^\dagger$ not allowed? What is the condition $\hat{U}$ must satisfy in order to preserve the canonical commutation relation $\left[\hat{Q},\hat{P}\right] = [\hat{q},\hat{p}] = i\hbar$?

(iv) Consider $\hat{U} = e^{\frac{i}{\hbar}\alpha(\hat{p})}$. Working in momentum representation, i.e. $\hat{p} \to p$ and $\hat{q} \to i\hbar\frac{\partial}{\partial p}$ acting on some wave-function $\psi(p)$, find how $\hat{q}$ and $\hat{p}$ are transformed under $\hat{U}$.

Solution

(i) As suggested in the hint we notice that

$$\langle p_1\,|\,[\hat{q},\hat{p}]\,|\,p_2\rangle = i\hbar\;\delta(p_1 - p_2) = -(p_1 - p_2)\,\langle p_1\,|\,\hat{q}\,|\,p_2\rangle$$

This relation suggests that the the solution is of the form

$$\langle p_1\,|\,\hat{q}\,|\,p_2\rangle = F(p_1, p_2)$$

where $F(p_1, p_2)$ is a distribution that satisfies

$$-(p_1 - p_2)F(p_1, p_2) = i\hbar\;\delta(p_1 - p_2)$$

Since we are dealing with distributions we cannot divide both sizes by $(p_1 - p_2)(x^{-1}\delta(x)$ is undefined). By defining $p \equiv p_1 - p_2$ and using the Fourier representation of the δ-function $\delta(q) = \frac{1}{2\pi}\int dp\; e^{iqp}$ we find that

$$i\hbar = -\int dp\; pF(p_1, p_1 - p)e^{-ipq} = -i\frac{\partial}{\partial q}\int dp\; F(p_1, p_1 - p)e^{-ipq}$$

Integrating over q we find

$$-(\hbar q + C(p_1)) = \int dp\; F(p_1, p_1 - p)e^{-ipq}$$

where $C(p_1)$ is a function to be determined. Now applying the inverse Fourier transform:

$$F(p_1, p_2) = -\frac{1}{2\pi}\int dq(\hbar q + C(p_1))e^{ipq} = -\left[-i\hbar\frac{\partial}{\partial p_1} + C(p_1)\right]\delta(p_1 - p_2)$$

Therefore, the result is

$$\langle p_1\,|\,\hat{q}\,|\,p_2\rangle = i\hbar\delta'(p_1 - p_2) - C(p_1)\delta(p_1 - p_2)$$

(ii) If we replace the operator $\hat{q}$ by $\hat{q} + c(\hat{p})$, where c is an arbitrary function, the matrix element $\langle p_1\,|\,\hat{q}\,|\,p_2\rangle$ would change by the term $c(p_1)\delta(p_1 - p_2)$.

Therefore, we could redefine the operator $\hat{q}$ to remove the term proportional to $\delta(p_1 - p_2)$ in the result, so as to obtain $\langle p_1\,|\,\hat{q}\,|\,p_2\rangle = i\hbar\delta'(p_1 - p_2)$.

The meaning of this is that the function $C(p_1)$ cannot be found from the commutation relation alone. In fact, the above transformation of $\hat{q}$ still preserves the commutation relation. As a consequence, the representations for $\hat{q}$ and $\hat{p}$ are defined up to a function.

(iii) It is not allowed since $\hat{Q}$ (or $\hat{P}$) must be an hermitian operator. We have

$$\hat{Q}^\dagger = \left(\hat{U}\hat{q}\hat{V}\right)^\dagger = \hat{V}^\dagger\hat{q}\hat{U}^\dagger \neq \hat{Q} \quad \text{if } \hat{V} \neq \hat{U}^\dagger$$

Furthermore

$$\left[\hat{Q}, \hat{P}\right] = \left[\hat{U}\hat{q}\hat{U}^\dagger, \hat{U}\hat{p}\hat{U}^\dagger\right] = \hat{U}\hat{q}\hat{U}^\dagger\hat{U}\hat{p}\hat{U}^\dagger - \hat{U}\hat{p}\hat{U}^\dagger\hat{U}\hat{q}\hat{U}^\dagger$$

If we take $\hat{U}$ as unitary operator, i.e. $\hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = \mathbb{1}$, then

$$\hat{U}\hat{q}\hat{U}^\dagger\hat{U}\hat{p}\hat{U}^\dagger - \hat{U}\hat{p}\hat{U}^\dagger\hat{U}\hat{q}\hat{U}^\dagger = \hat{U}\left[\hat{q}, \hat{p}\right]\hat{U}^\dagger = \hat{U}i\hbar\hat{U}^\dagger = i\hbar$$

(iv) To see how $\hat{q}$ and $\hat{p}$ transform we must apply the new operators to some wave-function $\psi(p)$. It is easy to see that $\hat{p}$ doesn't transform, since it commutes with $\hat{U}$. About the transformation of $\hat{q}$:

$$\hat{Q}\,|\,\psi\rangle = \hat{U}\hat{q}\hat{U}^\dagger\,|\,\psi\rangle = e^{\frac{i}{\hbar}\alpha(\hat{p})}\hat{q}e^{-\frac{i}{\hbar}\alpha(\hat{p})}\,|\,\psi\rangle \to e^{\frac{i}{\hbar}\alpha(p)}i\hbar\frac{\partial}{\partial p}\left(e^{-\frac{i}{\hbar}\alpha(p)}\psi(p)\right) = \left[i\hbar\frac{\partial}{\partial p} + \frac{\partial\alpha(p)}{\partial p}\right]\psi(p)$$

Therefore we see that:

$$i\hbar\frac{\partial}{\partial p} \longrightarrow i\hbar\frac{\partial}{\partial p} + \frac{\partial\alpha(p)}{\partial p}$$

that is equivalent to redefine $\hat{q}$ as $\hat{Q} = \hat{q} + c(\hat{p})$, with $c(\hat{p})$ defined by $\alpha(p)$ as above.

Exercise 21

Consider the following Hamilton operator

$$\hat{H} = \frac{\hat{p}^2}{2} + \cosh\hat{x} \tag{1}$$

Write the time independent Schrödinger equation

$$\hat{H}\,|\psi_n\rangle = E_n\,|\psi_n\rangle \tag{2}$$

in components in position and momentum representation.

Solution

(i) **The position representation**
Let $\psi_n(x) = \langle x|\psi_n\rangle$. Acting with $\langle x|$ on the equation we obtain

$$\begin{aligned}\left\langle x|\frac{\hat{p}^2}{2}|\psi_n\right\rangle + \cosh x\psi_n(x) &= E_n\psi_n(x)\\ -\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2}\psi_n(x) + \cosh x\psi_n(x) &= E_n\psi_n(x)\end{aligned} \tag{3}$$

(ii) **The momentum representation**
Let $\psi_n(p) = \langle p|\psi_n\rangle$. Acting with $\langle p|$ on the equation we obtain

$$\begin{aligned}&\left\langle p|\frac{\hat{p}^2}{2}|\psi_n\right\rangle + \langle p|\cosh x|\psi_n\rangle = E_n\psi_n(p)\\ &\cosh\hat{x} = \frac{e^{\hat{x}} + e^{-\hat{x}}}{2}\\ &\text{In momentum representation}\quad \hat{x} = i\hbar\frac{\partial}{\partial p}\\ &\text{Combining this with the result from the last problem sheet we obtain}\\ &\frac{p^2}{2}\psi_n(p) + \frac{\psi_n(p+i\hbar) + \psi_n(p-i\hbar)}{2} = E_n\psi_n(p)\end{aligned} \tag{4}$$

Exercise 22 (Central tutorial)

Let us consider a harmonic oscillator of unit mass, described by the Lagrangian

$$L = \frac{\dot{q}^2}{2} - \frac{\omega^2}{2} q^2 \tag{5}$$

where ω is the constant frequency.

(i) What is the corresponding Hamiltonian? Now promote coordinate and momenta into operators. What is the commutation relation which they should satisfy? Set $\hbar = 1$.

(ii) Let's define creation and annihilation operators respectively as

$$\hat{a}^\dagger = \sqrt{\frac{\omega}{2}} \left(\hat{q} - \frac{i}{\omega} \hat{p} \right) \qquad \hat{a} = \sqrt{\frac{\omega}{2}} \left(\hat{q} + \frac{i}{\omega} \hat{p} \right) \tag{6}$$

Derive the commutation relation which they should satisfy. Express the Hamiltonian in terms of the creation and annihilation operators.

(iii) What is the energy of the ground state? Calculate $\left\langle (\Delta\hat{q})^2 \right\rangle_0 \left\langle (\Delta\hat{p})^2 \right\rangle_0$, where for an operator $\hat{A}$ we have $\left\langle \hat{A} \right\rangle_0 = \langle 0 | \hat{A} | 0 \rangle$, $\hat{a} |0\rangle = 0$ and $\Delta\hat{A} = \sqrt{\left\langle \hat{A}^2 \right\rangle - \left\langle \hat{A} \right\rangle^2}$.

(iv) Now let λ, μ be complex constants. Show the following identities:

$$\begin{aligned} \left[\hat{a}, e^{\lambda \hat{a}^\dagger} \right] &= \lambda e^{\lambda \hat{a}^\dagger} \\ e^{\lambda \hat{a}} e^{\mu \hat{a}^\dagger} &= e^{\mu\lambda} e^{\mu \hat{a}^\dagger} e^{\lambda \hat{a}} \end{aligned} \tag{7}$$

Hint: Consider the function

$$f(\lambda, \mu) = e^{\lambda \hat{a}} e^{\mu \hat{a}^\dagger} e^{-\lambda \hat{a}} e^{-\mu \hat{a}^\dagger} \tag{8}$$

and its derivatives $\frac{\partial f}{\partial \lambda}, \frac{\partial f}{\partial \mu}$.

Solution

(i) First, the momentum is given by

$$p = \frac{\partial L}{\partial \dot{q}} = \dot{q} \tag{9}$$

Then one can easily see that the Hamiltonian is given by

$$H = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} \tag{10}$$

After we promote p and q into operators, they should satisfy the following commutation relation: $[\hat{q}, \hat{p}] = i\hat{I}$.

(ii) First, the commutation relation is given by

$$\left[\hat{a}, \hat{a}^\dagger \right] = \frac{\omega}{2} \left[\hat{q} + \frac{i}{\omega} \hat{p}, \hat{q} - \frac{i}{\omega} \hat{p} \right] = \hat{I} \tag{11}$$

where in the last step we have used the commutation relation of coordinate and momentum. From the definition of creation and annihilation operators we see that

$$\hat{q} = \frac{1}{\sqrt{2\omega}} \left(\hat{a} + \hat{a}^\dagger \right) \qquad \hat{p} = i\sqrt{\frac{\omega}{2}} \left(\hat{a}^\dagger - \hat{a} \right) \tag{12}$$

We can use this to obtain the Hamiltonian:

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{\omega^2 \hat{q}^2}{2} = \frac{\omega}{2} \left(\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger \right) = \frac{\omega}{2} \left(2\hat{a}^\dagger \hat{a} + \hat{I} \right) \tag{13}$$

where in the last step we have used the commutation relation of creation and annihilation operators.

(iii) Clearly, the energy of the ground state is given by $E_0 = \frac{\omega}{2}$.

Now remember that the uncertainty of an operator, $\Delta\hat{A}$ is defined by $\Delta\hat{A} = \sqrt{\left\langle \hat{A}^2 \right\rangle - \left\langle \hat{A} \right\rangle^2}$.
Then we have

$$\begin{aligned} &\left\langle \hat{q}^2 \right\rangle_0 = \langle 0| \frac{\hat{a}\hat{a}^\dagger}{2\omega} |0\rangle = \frac{1}{2\omega} \quad , \quad \langle \hat{q} \rangle_0 = 0 \implies \left\langle (\Delta\hat{q})^2 \right\rangle_0 = \frac{1}{2\omega} \\ &\left\langle \hat{p}^2 \right\rangle_0 = \frac{\omega}{2} \quad , \quad \langle \hat{p} \rangle_0 = 0 \implies \left\langle (\Delta\hat{p})^2 \right\rangle_0 = \frac{\omega}{2} \\ &\text{So we have: } \left\langle (\Delta\hat{q})^2 \right\rangle_0 \left\langle (\Delta\hat{p})^2 \right\rangle_0 = \frac{1}{4} \end{aligned} \tag{14}$$

(iv) **First relation:** First we will show by induction $\left[\hat{a}, (\hat{a}^\dagger)^k\right] = k(\hat{a}^\dagger)^{k-1}$.
Cases k=0,1 are trivial due to the commutation relations.
From k to k+1 we have

$$\begin{aligned} &\left[\hat{a}, (\hat{a}^\dagger)^{k+1}\right] = \left[\hat{a}, (\hat{a}^\dagger)^k \hat{a}^\dagger\right] = |\textit{by the properties of the commutator}| = (\hat{a}^\dagger)^k + \left[\hat{a}, (\hat{a}^\dagger)^k\right] = \\ &= |\textit{using the relation for } k| = (1+k)(\hat{a}^\dagger)^k \end{aligned} \tag{15}$$

So it follows that

$$\left[\hat{a}, e^{\lambda\hat{a}^\dagger}\right] = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left[\hat{a}, (\hat{a}^\dagger)^k\right] = \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \left[\hat{a}, (\hat{a}^\dagger)^k\right] = \sum_{k=1}^{\infty} \frac{\lambda^k k}{k!} (\hat{a}^\dagger)^{k-1} = \lambda e^{\lambda\hat{a}^\dagger} \tag{16}$$

Second relation: Following the hint and using previously proved relation we obtain

$$\begin{aligned} &\frac{\partial f}{\partial \lambda} = e^{\lambda\hat{a}} \hat{a} e^{\mu\hat{a}^\dagger} e^{-\lambda\hat{a}} e^{-\mu\hat{a}^\dagger} - e^{\lambda\hat{a}} e^{\mu\hat{a}^\dagger} \hat{a} e^{-\lambda\hat{a}} e^{-\mu\hat{a}^\dagger} = \mu f \\ &\frac{\partial f}{\partial \mu} = e^{\lambda\hat{a}} e^{\mu\hat{a}^\dagger} \hat{a}^\dagger e^{-\lambda\hat{a}} e^{-\mu\hat{a}^\dagger} - e^{\lambda\hat{a}} e^{\mu\hat{a}^\dagger} e^{-\lambda\hat{a}} \hat{a}^\dagger e^{-\mu\hat{a}^\dagger} = \lambda f \\ &\textit{Note that for the last relation we have } \left[e^{-\lambda\hat{a}}, \hat{a}^\dagger\right] = -\lambda e^{-\lambda\hat{a}} \end{aligned} \tag{17}$$

This means that $f(\lambda, \mu) = Ae^{\lambda\mu}$. Since $f(0,0) = 1$, we can set $A = 1$. Then we have

$$e^{\lambda\hat{a}} e^{\mu\hat{a}^\dagger} e^{-\lambda\hat{a}} e^{-\mu\hat{a}^\dagger} = e^{\lambda\mu} \implies e^{\lambda\hat{a}} e^{\mu\hat{a}^\dagger} = e^{\lambda\mu} e^{\mu\hat{a}^\dagger} e^{\lambda\hat{a}} \tag{18}$$

Exercise 23 (Central tutorial)

We define the coherent state as an eigenstate of the annihilation operator:

$$\hat{a}\,|a\rangle = a\,|a\rangle \tag{19}$$

(i) Calculate using the results of the previous exercise $\left\langle (\Delta\hat{q})^2 \right\rangle_a \left\langle (\Delta\hat{p})^2 \right\rangle_a$. How can we interpret this result?

(ii) Prove the following relation

$$|a\rangle = e^{a\hat{a}^\dagger}\,|0\rangle \tag{20}$$

where

$$\hat{a}\,|0\rangle = 0 \quad \text{and} \quad |a\rangle = \sum_n \frac{a^n}{\sqrt{n!}}\,|n\rangle \quad \text{with} \quad |n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\,|0\rangle\,. \tag{21}$$

(iii) Show that the states $|n\rangle$ are orthonormal eigenvectors of the occupation number operator $\hat{N} = \hat{a}^\dagger\hat{a}$.

(iv) Find $\psi_n(a) = \langle a|n\rangle$ for eigenvectors of $\hat{N}$.

Solution

(i) Using the definition of creation and annihilation operators from the previous problem we have

$$\begin{aligned}
&\langle \hat{q} \rangle_a = \langle a|\hat{q}|a\rangle = \frac{1}{\sqrt{2\omega}} \left\langle a|\hat{a}+\hat{a}^\dagger|a\right\rangle = \frac{1}{\sqrt{2\omega}}(a+a^*) \langle a \,|\, a\rangle \\
&\left\langle (\hat{q})^2 \right\rangle_a = \frac{1}{2\omega} \left\langle a|(\hat{a}+\hat{a}^\dagger)(\hat{a}+\hat{a}^\dagger)|a\right\rangle = \frac{1}{2\omega} \left\langle a|\hat{a}^2 + (\hat{a}^\dagger)^2 + 2\hat{a}^\dagger\hat{a} + 1|a\right\rangle = \frac{1}{2\omega} \langle a \,|\, a\rangle + \frac{1}{2\omega}(a+a^*)^2 \langle a \,|\, a\rangle \\
&\left\langle (\Delta\hat{q})^2 \right\rangle_a = \frac{1}{2\omega} \\
&\textit{Similarly} \quad \langle \hat{p} \rangle_a = i\sqrt{\frac{\omega}{2}}(a^*-a) \langle a \,|\, a\rangle \quad \left\langle \hat{p}^2 \right\rangle_a = \frac{\omega}{2} \langle a \,|\, a\rangle - \frac{\omega}{2}(a^*-a)^2 \langle a \,|\, a\rangle \\
&\left\langle (\Delta\hat{p})^2 \right\rangle_a = \frac{\omega}{2} \\
&\implies \left\langle (\Delta\hat{q})^2 \right\rangle_a \left\langle (\Delta\hat{p})^2 \right\rangle_a = \frac{1}{4}(\langle a \,|\, a\rangle)^2 \implies (\Delta\hat{p})^2(\Delta\hat{q})^2 = \frac{1}{4}
\end{aligned} \tag{22}$$

So we see that the coherent states minimise the uncertainty in both coordinate and momenta.

(ii)

$$|a\rangle = \sum_n \frac{a^n}{\sqrt{n!}} |n\rangle = \sum_n \frac{a^n}{n!} (\hat{a}^\dagger)^n |0\rangle = e^{a\hat{a}^\dagger} |0\rangle \tag{23}$$

(iii) Suppose that $n_1 \neq n_2$. Then the bracket is vanishing because we will always have one creation operator action on bra or one annihilation operator acting on ket after we apply the their commutation relation. As a simple example observe the following

$$\begin{aligned}
&\langle n_1|n_2\rangle = \frac{1}{\sqrt{n_1!n_2!}} \left\langle 0|\hat{a}^{n_1}(\hat{a}^\dagger)^{n_2}|0\right\rangle \quad \text{set } n_1 = 1 \text{ and } n_2 = 2 \\
&\frac{1}{\sqrt{2}} \left\langle 0|\hat{a}\hat{a}^\dagger\hat{a}^\dagger|0\right\rangle = \frac{1}{\sqrt{2}} \left\langle 0|\hat{a}^\dagger\hat{a}\hat{a}^\dagger|0\right\rangle + \left\langle 0|\hat{a}^\dagger|0\right\rangle = 0
\end{aligned} \tag{24}$$

Now set $n_1 = n_2 = n$. Let's prove by induction that $\left\langle 0|\hat{a}^n(\hat{a}^\dagger)^n|0\right\rangle = n!$.
For $n = 0, 1$ the relation is true. Then replacing n by $n+1$ we have

$$\begin{aligned}
\left\langle 0|\hat{a}^{n+1}(\hat{a}^\dagger)^{n+1}|0\right\rangle &= \left\langle 0|\hat{a}^n\hat{a}(\hat{a}^\dagger)^{n+1}|0\right\rangle = \left\langle 0|\hat{a}^n \left[\hat{a}, (\hat{a}^\dagger)^{n+1}\right]|0\right\rangle + \left\langle 0|\hat{a}^n(\hat{a}^\dagger)^{n+1}\hat{a}|0\right\rangle = \\
&= (n+1) \left\langle 0|\hat{a}^n(\hat{a}^\dagger)^n|0\right\rangle = (n+1)!
\end{aligned} \tag{25}$$

where in the second line we have used the relation from problem 22. Hence $\langle n|n\rangle = 1$.
Now we have

$$\hat{N}|n\rangle = \frac{\hat{a}^\dagger\hat{a}(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle = \frac{\hat{a}^\dagger \left[\hat{a}, (\hat{a}^\dagger)^n\right]}{\sqrt{n!}} |0\rangle + 0 = \frac{n(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle = n|n\rangle \tag{26}$$

(iv)

$$\psi_n(a) = \langle a|n\rangle = \sum_i \frac{a^i}{\sqrt{i!}} \langle i|n\rangle = \frac{a^n}{\sqrt{n!}} \tag{27}$$

Exercise 24

Consider the Pauli matrices $\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3$.

(i) Show that $\hat{\sigma}_1\hat{\sigma}_2 = i\hat{\sigma}_3$. What are $\hat{\sigma}_1\hat{\sigma}_3$ and $\hat{\sigma}_2\hat{\sigma}_3$?

(ii) Show that $[\hat{\sigma}_1, \hat{\sigma}_2] = 2i\hat{\sigma}_3$ and $\{\hat{\sigma}_1, \hat{\sigma}_2\} = 0$.

The properties above can be generalized and encoded in the following relations:

$$\hat{\sigma}_i\hat{\sigma}_j = \delta_{ij}\mathbb{1} + i\epsilon_{ij}{}^k\hat{\sigma}_k \quad , \quad \{\hat{\sigma}_i, \hat{\sigma}_j\} = 2\delta_{ij} \quad , \quad [\hat{\sigma}_i, \hat{\sigma}_j] = 2i\epsilon_{ij}{}^k\hat{\sigma}_k$$

where $\epsilon_{ij}{}^k$ indicates the Levi-Civita tensor (or *totally anti-symmetric* tensor).

(iii) Let v^j be three components of a (real) vector, define the spin operator as $\hat{\sigma} = (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$ and let

$$v \cdot \hat{\sigma} \equiv v^j \hat{\sigma}_j = v^1 \hat{\sigma}_1 + v^2 \hat{\sigma}_2 + v^3 \hat{\sigma}_3$$

Calculate $(v \cdot \hat{\sigma})^2$ and show that

$$\exp(iv \cdot \hat{\sigma}) = \cos|v| + i\frac{v \cdot \hat{\sigma}}{|v|} \sin|v|$$

where $|v| = \sqrt{v^j v_j}$.

Solution Recall that the Pauli matrices are:

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(i)

$$\hat{\sigma}_1 \hat{\sigma}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = i\hat{\sigma}_3$$

Similarly:

$$\hat{\sigma}_1 \hat{\sigma}_3 = -i\hat{\sigma}_2 \quad , \quad \hat{\sigma}_2 \hat{\sigma}_3 = i\hat{\sigma}_1$$

(ii) By explicit matrix calculations you can see

$$[\hat{\sigma}_1, \hat{\sigma}_2] = \hat{\sigma}_1 \hat{\sigma}_2 - \hat{\sigma}_2 \hat{\sigma}_1 = 2i\hat{\sigma}_3 \quad , \quad \{\hat{\sigma}_1, \hat{\sigma}_2\} = \hat{\sigma}_1 \hat{\sigma}_2 + \hat{\sigma}_2 \hat{\sigma}_1 = 0$$

(iii)

$$(v \cdot \hat{\sigma})^2 = v^i v^j \hat{\sigma}_i \hat{\sigma}_j = v^i v^j \left(\delta_{ij} \mathbb{1} + i{\epsilon_{ij}}^k \hat{\sigma}_k\right) \overset{\text{symmetry}}{=} v^i v^j \delta_{ij} \mathbb{1} = |v|^2 \mathbb{1}$$

To calculate the exponential we can use the previous result.

$$\exp(iv \cdot \hat{\sigma}) = \sum_{k=0}^{\infty} \frac{i^k}{k!} (v \cdot \hat{\sigma})^k = \sum_{k=0}^{\infty} \frac{i^{2k}}{(2k)!} (v \cdot \hat{\sigma})^{2k} + \sum_{k=0}^{\infty} \frac{i^{2k+1}}{(2k+1)!} (v \cdot \hat{\sigma})^{2k+1} =$$

$$= \mathbb{1} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (|v|)^{2k} + i\frac{(v \cdot \hat{\sigma})}{|v|} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (|v|)^{2k+1} =$$

$$= \mathbb{1} \cos|v| + i\frac{(v \cdot \hat{\sigma})}{|v|} \sin|v|$$

General information

The *lecture* takes place on:
  Monday at 10:00 - 12:00 c.t. in B 052 (Theresienstraße 37)
  Friday at 10:00 - 12:00 c.t. in B 052 (Theresienstraße 37)
The *central tutorial* takes place on Monday at 12:00 - 14:00 c.t. in B 139 (Theresienstraße 37)
The *webpage* for the lecture and exercises can be found at

`https://www.physik.uni-muenchen.de/lehre/vorlesungen/wise_19_20/T_M1_TV_-Quantum-Mechanics-II`