Prof. Dr. Viatcheslav Mukhanov Winter term 2019/2020

Exercises on Quantum Mechanics II (TM1/TV)

Solution 1, discussed October 21 - October 25, 2019

Exercise 1

Recall that in a vector space $\mathcal{V}$ the following axioms have to be fulfilled:
Let $|\psi\rangle, |\phi\rangle \in \mathcal{V}$ and $a, b \in \mathbb{C}$, then:
1) $a(|\psi\rangle + |\phi\rangle) = a|\psi\rangle + a|\phi\rangle$ and $(a+b)|\psi\rangle = a|\psi\rangle + b|\psi\rangle$
2) $a(b|\psi\rangle) = b(a|\psi\rangle)$
3) $\exists \mathbf{0} \in \mathcal{V}$ such that $\forall |\psi\rangle \in \mathcal{V}$, $0 \cdot |\psi\rangle = \mathbf{0}$
4) $1 \cdot |\psi\rangle = |\psi\rangle$
Show that:

(i) $|\psi\rangle + \mathbf{0} = |\psi\rangle$

(ii) $a(|\phi\rangle - |\psi\rangle) = a|\phi\rangle - a|\psi\rangle$

Solution:

(i) $|\psi\rangle + \mathbf{0} = |\psi\rangle + 0|\psi\rangle \overset{1)}{=} (1+0)|\psi\rangle = 1|\psi\rangle = |\psi\rangle$

(ii) $a(|\phi\rangle + \beta|\psi\rangle) \overset{1)}{=} a|\phi\rangle + a\beta|\psi\rangle \overset{2)}{=} a|\phi\rangle + \beta a|\psi\rangle$
Now setting $\beta = -1$ gives the expression in ii).

Exercise 2

Let $\mathcal{V}$ be a $\mathbb{C}$ vector space.

(i) What are the necessary conditions on all $|\psi\rangle \in \mathcal{V}$ for $\mathcal{V}$ to be a Hilbert space?

(ii) Now let $|\psi\rangle$ be an ordered sequence of complex numbers ψ_i, where $\sum_i |\psi_i|^2 < \infty$. Is it possible to define a scalar product $(\bullet, \bullet)$ according to $(|\psi\rangle, |\phi\rangle) = \sum_i \psi_i^* \phi_i$?

(iii) Is it possible to define a hermitian scalar product $(\bullet, \bullet)$ according to $(|\psi\rangle, |\phi\rangle) = \sum_i \psi_i \phi_i$?

Solution:

(i) There exists a map $(\bullet, \bullet) : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ called inner product with the properties $\forall |\psi\rangle \in \mathcal{V}$:

$$1)(|\phi\rangle, |\psi\rangle) = (|\psi\rangle, |\phi\rangle)^*$$
$$2)(|\phi\rangle, \alpha|\psi\rangle + \beta|\chi\rangle) = \alpha(|\phi\rangle, |\psi\rangle) + \beta(|\phi\rangle, |\chi\rangle) \text{ (linearity)}$$
$$3) \||\psi\rangle\| := \sqrt{(|\psi\rangle, |\psi\rangle)} \geq 0$$
$$\||\psi\rangle\| = 0 \Leftrightarrow |\psi\rangle = \mathbf{0} \text{ (positive definiteness)}$$

If $\mathcal{V}$ is complete with respect to $\|\bullet\|$ (i.e. every Cauchy sequence in $\mathcal{V}$ converges in $\mathcal{V}$), $\mathcal{V}$ is a Hilbert space.

(ii) Yes (easy to check for the axioms).

(iii) No because of the first axiom.

Exercise 3 (central tutorial)

Take the set $L^2(a,b)$ of all complex square integrable functions on the interval $[a,b]$ and the canonical addition map and scalar product as:
1) $\psi, \phi \in L^2(a,b)$: $(\psi + \phi)(x) := \psi(x) + \phi(x)$
2) $(\psi, \phi) := \int_a^b \psi^*(x)\phi(x)\mathrm{d}x$
Show that the emerging space $\mathcal{T}_F$ is a Hilbert space.

Solution:

To show the vector space axioms we have to show that $\forall f, g \in L^2(a,b)$ and $\alpha \in \mathbb{C}$ also $\alpha f \in L^2(a,b)$ and $f + g \in L^2(a,b)$.
$\int_a^b |\alpha f|^2 = |\alpha|^2 \int_a^b |f|^2 < \infty$ as $\int_a^b |f|^2 < \infty$ and $|\alpha| < \infty$
$\int_a^b |f+g|^2 = \int_a^b (|f|^2 + |g|^2 + 2\mathfrak{Re}(fg^*)) \leq \int_a^b (|f|^2 + |g|^2 + 2\,|fg^*|) = \int_a^b (|f|^2 + |g|^2 + 2\,|f|\,|g|)$
$\leq \int_a^b (|f|^2 + |g|^2 + |f|^2 + |g|^2 = 2\int_a^b |f|^2 + 2\int_a^b |g|^2 < \infty$
The properties of the inner product are fulfilled by definition of the Lebesgue integral.

Exercise 4 (central tutorial)

Let $\mathcal{H}$ and $\mathcal{H}^*$ be Hilbert spaces dual to each other. A bra vector $\langle\phi| \in \mathcal{H}^*$ corresponding to a ket vector $|\phi\rangle \in \mathcal{H}$ is defined via the scalar product as $\langle\phi|\chi\rangle \equiv (|\phi\rangle, |\chi\rangle)$.
Take $|\Psi\rangle = a\,|\phi\rangle + b\,|\psi\rangle \in \mathcal{H}$ and $\langle\Phi| = a^*\,\langle\phi| + b^*\,\langle\psi| \in \mathcal{H}^*$. Show that $\langle\Phi|$ corresponds to $|\Psi\rangle$, i.e. $\langle\Phi| = \langle\Psi|$.

Solution:

We have to show that $(|\Psi\rangle, |\chi\rangle) = \langle\Phi|\chi\rangle$.

$$\begin{aligned}(|\Psi\rangle, |\chi\rangle) &= (|\chi\rangle, |\Psi\rangle)^* = (|\chi\rangle, a\,|\phi\rangle + b\,|\psi\rangle)^* = a^*(|\chi\rangle, |\phi\rangle)^* + b^*(|\chi\rangle, |\psi\rangle)^* = a^*(|\phi\rangle, |\chi\rangle) + b^*(|\psi\rangle, |\chi\rangle) \\ &= a^*\,\langle\phi|\chi\rangle + b^*\,\langle\psi|\chi\rangle = \langle\Phi|\chi\rangle \\ &\Rightarrow \langle\Phi| = \langle\Psi|\end{aligned}$$

Exercise 5

Let $\mathcal{H}$ be a Hilbert space with respect to the inner product $\langle\bullet|\bullet\rangle$. Prove the Cauchy-Schwartz inequality $|\langle\psi|\phi\rangle| \leq \||\psi\rangle\|\,\||\phi\rangle\|$, where $\||\psi\rangle\| = \sqrt{\langle\psi|\psi\rangle}$.

Solution:

Define: $|a\rangle = |\psi\rangle + a\,|\phi\rangle$
By positive definiteness of the inner product we know: $\langle a|a\rangle = \langle\psi|\psi\rangle + a\,\langle\psi|\phi\rangle + a^*\,\langle\phi|\psi\rangle + |a|^2\,\langle\phi|\phi\rangle \geq 0$
Choose $a = -\frac{\langle\phi|\psi\rangle}{N}$, then $\langle\psi|\psi\rangle + \frac{|\langle\phi|\psi\rangle|^2}{N^2}\,\langle\phi|\phi\rangle \geq 2\frac{|\langle\phi|\psi\rangle|^2}{N}$
Choose $N = \langle\phi|\phi\rangle$, then $\langle\psi|\psi\rangle \geq \frac{|\langle\phi|\psi\rangle|^2}{\langle\phi|\phi\rangle} \quad \Leftrightarrow |\langle\phi|\psi\rangle| \leq \||\psi\rangle\|\,\||\phi\rangle\|$

Exercise 6 (central tutorial)

The set $\{|\phi_i\rangle\}$ forms a basis of the Hilbert space $\mathcal{H}$. Prove the sequence

$$|\chi_1\rangle = \frac{|\phi_1\rangle}{\||\phi_1\rangle\|}, \quad |\chi_k\rangle = \frac{|\phi_k\rangle - \sum_{j=1}^{k-1}\langle\chi_j|\phi_k\rangle\,|\chi_j\rangle}{\left\||\phi_k\rangle - \sum_{j=1}^{k-1}\langle\chi_j|\phi_k\rangle\,|\chi_j\rangle\right\|} \quad \forall k > 1$$

forms an orthonormal basis of $\mathcal{H}$.

Solution:

Proof by induction that $\langle \chi_i | \chi_j \rangle = \delta_{ij}$:

$$\langle \chi_k | \chi_k \rangle = \frac{\left\| |\phi_k\rangle - \sum_{j=1}^{k-1} \langle \chi_j | \phi_k \rangle |\chi_j\rangle \right\|^2}{\left\| |\phi_k\rangle - \sum_{j=1}^{k-1} \langle \chi_j | \phi_k \rangle |\chi_j\rangle \right\|^2} = 1$$

$$\begin{aligned}\langle \chi_1 | \chi_2 \rangle &= \frac{1}{\| |\phi_1\rangle \| \, \| |\phi_2\rangle - |\chi_1\rangle \langle \chi_1 | \phi_2 \rangle \|} \left(\langle \phi_1 | \phi_2 \rangle - \frac{\langle \phi_1 | \phi_1 \rangle}{\| |\phi_1\rangle \|^2} \langle \phi_1 | \phi_2 \rangle \right) \\ &= \frac{1}{\| |\phi_1\rangle \| \, \| |\phi_2\rangle - |\chi_1\rangle \langle \chi_1 | \phi_2 \rangle \|} \left(\langle \phi_1 | \phi_2 \rangle - \langle \phi_1 | \phi_2 \rangle \right) = 0\end{aligned}$$

Assuming $\langle \chi_1 | \chi_j \rangle = \delta_{1j}$ we show $\langle \chi_1 | \chi_{j+1} \rangle = \delta_{1,j+1}$:

$$\begin{aligned}\langle \chi_1 | \chi_{j+1} \rangle &= \frac{1}{\left\| |\phi_{j+1}\rangle - \sum_{k=1}^{j} |\chi_k\rangle \langle \chi_k | \phi_{j+1} \rangle \right\|} \langle \chi_1 | \left(|\phi_{j+1}\rangle - \sum_{k=1}^{j} |\chi_k\rangle \langle \chi_k | \phi_{j+1} \rangle \right) \\ &= \frac{1}{\left\| |\phi_{j+1}\rangle - \sum_{k=1}^{j} |\chi_k\rangle \langle \chi_k | \phi_{j+1} \rangle \right\|} \left(\langle \chi_1 | \phi_{j+1} \rangle - \sum_{k=1}^{j} \underbrace{\langle \chi_1 | \chi_k \rangle}_{\delta_{1k}} \langle \chi_k | \phi_{j+1} \rangle \right) \\ &= \frac{1}{\left\| |\phi_{j+1}\rangle - \sum_{k=1}^{j} |\chi_k\rangle \langle \chi_k | \phi_{j+1} \rangle \right\|} \left(\langle \chi_1 | \phi_{j+1} \rangle - \langle \chi_1 | \phi_{j+1} \rangle \right) = 0\end{aligned}$$

Assuming now that $\langle \chi_i | \chi_j \rangle = \delta_{ij}$ we show that $\langle \chi_{i+1} | \chi_j \rangle = 0$ for $i + 1 < j$:

$$\begin{aligned}\langle \chi_{i+1} | \chi_j \rangle &= \frac{1}{\| ... \|} \left(\langle \phi_{i+1} | - \sum_{l=1}^{i} \langle \phi_{i+1} | \chi_l \rangle \langle \chi_l | \right) |\chi_j\rangle = \frac{1}{\| ... \|} \left(\langle \phi_{i+1} | \chi_j \rangle - \sum_{l=1}^{i} \langle \phi_{i+1} | \chi_l \rangle \underbrace{\langle \chi_l | \chi_j \rangle}_{=\delta_{lj}} \right) \\ &= \frac{1}{\| ... \|} \left(\langle \phi_{i+1} | \chi_j \rangle - \langle \phi_{i+1} | \chi_j \rangle \right) = 0\end{aligned}$$

Therefore $\{|\chi_i\rangle\}$ is an orthogonal basis of $\mathcal{H}$.

Exercise 7

Let $\mathcal{T}_F$ be the Hilbert space of square integrable complex functions defined in Exercise 3.

(i) Which of the following operators are linear operators?

$$\hat{A} = (\bullet)^2, \hat{B} = \frac{\mathrm{d}}{\mathrm{d}x}(\bullet), \hat{C} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}(\bullet), \hat{D} = g(x)(\bullet), \hat{E} = g(x)(\bullet)^3$$

(ii) Which of the following symbols can be interpreted as operators on $\mathcal{T}_F$?

a) $\int \psi(x)(\bullet)\mathrm{d}x$

b) $\int (\bullet)k(x,y)(*)\mathrm{d}x\mathrm{d}y$

c) $\int k(x,y)(\bullet)\mathrm{d}x$

Solution:

(i) A linear operator $\hat{A}$ is defined as $\forall |\psi\rangle, |\phi\rangle \in \mathcal{H}$ and $a, b \in \mathbb{C}$: $\hat{A}(a|\psi\rangle + b|\phi\rangle) = a\hat{A}|\psi\rangle + b\hat{A}|\phi\rangle$. Therefore it is easy to check, that only $\hat{B}, \hat{C}$ and $\hat{D}$ are linear operators.

(ii) Only c) can be interpreted as an operator, as a) and b) don't return a function but a number.

Exercise 8

Let $\hat{A}, \hat{B}, \hat{C}$ be linear operators acting on a Hilbert space. Prove

$$\hat{A}(\hat{B} + \hat{C}) = \hat{A}\hat{B} + \hat{A}\hat{C}$$

Solution:

$$\hat{A}(\hat{B}+\hat{C})\,|\psi\rangle = \hat{A}(\hat{B}\,|\psi\rangle + \hat{C}\,|\psi\rangle) = \hat{A}\hat{B}\,|\psi\rangle + \hat{A}\hat{C}\,|\psi\rangle = (\hat{A}\hat{B}+\hat{A}\hat{C})\,|\psi\rangle$$

In the first step the definition of addition of operators was used, in the second step linearity was used and in the last step the definition of addtion was used again.

General information

The *lecture* takes place on:
Monday at 10:00 - 12:00 c.t. in B 052 (Theresienstraße 37)
Friday at 10:00 - 12:00 c.t. in B 052 (Theresienstraße 37)
The *central tutorial* takes place on Monday at 12:00 - 14:00 c.t. in B 139 (Theresienstraße 37)
The *webpage* for the lecture and exercises can be found at

`https://www.physik.uni-muenchen.de/lehre/vorlesungen/wise_19_20/T_M1_TV_-Quantum-Mechanics-II`