Prof. Dr. Viatcheslav Mukhanov Winter term 2019/2020

Exercises on Quantum Mechanics II (TM1/TV)

Problem set 14, discussed February 3 - February 7, 2020

Exercise 76 - Central tutorial

Consider a Stern-Gerlach Experiment oriented in the z-direction connected to a detector. Before the measurement the detector is in the state $\left| \psi^A[0] \right\rangle$. If a silver atom with spin up $|\uparrow\rangle$ is measured the detector is in the state $\left| \psi^A[\uparrow] \right\rangle$ and and for spin down $|\downarrow\rangle$ the state of the device is $\left| \psi^A[\downarrow] \right\rangle$. Let the silver atom be in a general spin state $\left| \psi^S \right\rangle = \alpha\, |\uparrow\rangle + \beta\, |\downarrow\rangle$, with $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$.

(i) Write down the general state $\left| \Psi^1 \right\rangle$ of the whole system before and after the measurement.

(ii) Write the states as $|\uparrow\rangle = (1,0)^T$, $|\downarrow\rangle = (0,1)^T$,
$\left| \psi^A[\uparrow] \right\rangle = (1,0,0)^T$, $\left| \psi^A[0] \right\rangle = (0,1,0)^T$, $\left| \psi^A[\downarrow] \right\rangle = (0,0,1)^T$
and express $\left| \Psi^1 \right\rangle$ in that basis.

Now we want to repeat this experiment N times with silver atoms being in the same state $\left| \psi^S \right\rangle$. Define new states of the detector as $\left| \psi^A_+[n] \right\rangle$ and $\left| \psi^A_-[m] \right\rangle$ where n and m are the numbers of silver atoms measured with spin up and down respectively.

(iii) What is the corresponding spin state for $\left| \psi^A_+[1] \right\rangle \otimes \left| \psi^A_-[2] \right\rangle$ if $N = 3$?

(iv) Write the state of the whole system before and after the measurement and bring it in the normalized form

$$\left| \Psi^N \right\rangle = \sum_{n=0}^{N} c_n\, |\uparrow_1\rangle \otimes \ldots \otimes |\uparrow_n\rangle \otimes |\downarrow_{n+1}\rangle \otimes \ldots \otimes |\downarrow_N\rangle \otimes \left| \psi^A_+[n] \right\rangle \otimes \left| \psi^A_-[N-n] \right\rangle \tag{1}$$

(v) Find the expectation values of n and m in the limit $N \to \infty$. Interpret your result.
(*Hint:* Use the fact that the binomial distribution becomes the normal distribution in the large N limit.)

Exercise 77 - Central tutorial

Consider a gaussian wave packet as in exercise 67:

$$\psi(q,t) = N \exp\left[\alpha(t) + \frac{i}{\hbar} p_{cl}(t)(q - q_{cl}(t)) - \frac{1}{2\sigma(t)}(q - q_{cl}(t))^2\right]. \tag{2}$$

where $p_{cl}(t) = m \frac{d}{dt} q_{cl}(t)$.

(i) Determine the normalization coefficient N (assuming that $q_{cl}(t)$ is real).

(ii) Determine the expectation value of $\hat{q}$ in the sate $\psi(q,t)$ and show that in the same state

$$\Delta q^2 \equiv \langle (\hat{q} - \langle \hat{q} \rangle)^2 \rangle = \frac{|\sigma(t)|^2}{2\Re\sigma(t)}. \tag{3}$$

(iii) Find the expression for the wave function in the momentum space.

(iv) Using the previous result, evaluate the expectation value of $\hat{p}$ and Δp^2.

(v) Verify that the Heisenberg uncertainty relations are satisfied. For which values of $\sigma(t)$ do we get an equality?

(vi) Until now we considered the Gaussian packet purely kinematically, without any dynamics. Consider now the Hamiltonian of the form

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}) \tag{4}$$

Remember that in Exercise 61 it was shown that $q_{cl}(t)$ satisfies the classical Newton's equations

$$\dot{p}_{cl}(t) = -V'(q_{cl}(t)) \tag{5}$$

while the function $\sigma^2(t)$ satisfied equation

$$\frac{d}{dt}\sigma^2(t) = \frac{i\hbar}{m} + \frac{i}{\hbar}V''(q_{cl}(t))\sigma^4(t). \tag{6}$$

Solve this equation for the free particle and the particle in a homogeneous (constant) force field.

(vii) Analyse the spreading behaviour of the wave packet in the momentum space.

(viii) Analyse the spreading behaviour of the wave packet in the position space.

Exercise 78

A Stern-Gerlach experiment oriented in the z-direction is fed with silver atoms. The result is that half of the atoms is measured with spin up and the other half with spin down.

(i) Write down the spin state of one silver atom.

(ii) The Stern-Gerlach experiment can be interpreted as a spin operator $\hat{S}_z$ with eigenstates $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$ and corresponding eigenvalues $\pm\frac{\hbar}{2}$. Write $\hat{S}_z, \hat{S}_x$ and $\hat{S}_y$ in terms of their eigenstates.

(iii) Use the fact that $|\uparrow_x\rangle$ measured in z-direction gives eigenvalues $\pm\frac{\hbar}{2}$ with $\frac{1}{2}$ probability but never $-\frac{\hbar}{2}$ if measured in x-direction to write $\hat{S}_x$ in terms of $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$. There should be one free parameter left.

(iv) Repeat the same for the y-direction.

(v) Use the fact that $|\uparrow_x\rangle$ measured in y-direction gives eigenvalues $\pm\frac{\hbar}{2}$ with $\frac{1}{2}$ probability to fix the remaining free parameter.

(vi) You now have $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ in the eigenbasis of $\hat{S}_z$. What do they remind you of?

Exercise 79

Assume that a system S is coupled to a measuring device M. The state before the measurement is

$$|\Psi_0\rangle = |s\rangle \otimes |M_0\rangle\,, \qquad |s\rangle = \sum_n a_n\,|s_n\rangle \tag{7}$$

and the state after the measurement is

$$|\Psi\rangle = \sum_n a_n\,|s_n\rangle \otimes |M_n\rangle \tag{8}$$

where the device states $|M_n\rangle$ are orthogonal. Find the density operator $\hat{\rho}$ for the subsystem S after the measurement.

Exercise 80

Consider the same physical system as in exercise 73.

(i) Show that the classical equation of motion for $\varphi(t)$ is

$$T\ddot{\varphi} = \dot{p_\varphi} = -g(t)p_q - Ig(t)^2\varphi. \tag{9}$$

(ii) Quantize the Hamiltonian found the last week (exercise 73). How does the equation of motion change after the quantization?

(iii) Assume that the driving current is constant, i.e. $g(t) = g_0$. Find the general (operator) solution of the equations of motion with the initial conditions $\hat{\varphi}(t=0) = \hat{\varphi}_0$ and $\dot{\hat{\varphi}}(t=0) = T^{-1}\hat{p}_\varphi^{(0)}$. It is useful to introduce the frequency $Ig_0^2 = T\omega^2$. The result you should get is

$$\hat{\varphi}(t) = -\frac{\hat{p}_q}{Ig_0} + \left(\hat{\varphi}_0 + \frac{\hat{p}_q}{Ig_0}\right)\cos(\omega t) + \frac{\hat{p}_\varphi^{(0)}}{T\omega}\sin(\omega t). \tag{10}$$

(iv) Verify the commutation relation

$$[\hat{\varphi}(t), \hat{\varphi}(t+\tau)] = \frac{i\hbar}{T\omega}\sin(\omega\tau). \tag{11}$$

Exercise 81

Consider a system of two spin $\frac{1}{2}$ particles. Let us denote $|\uparrow\rangle$ the normalized eigenvector of σ_3 with eigenvalue $+1$ and $|\downarrow\rangle$ the eigenvector with eigenvalue -1.

(i) Express the normalized eigenvectors of σ_1 $|\rightarrow\rangle$ (eigenvalue $+1$) and $|\leftarrow\rangle$ (eigenvalue -1) in terms of $|\uparrow\rangle$ and $|\downarrow\rangle$.

(ii) Find a vector in two particle Hilbert space

$$|\psi\rangle = \alpha_1 |\uparrow\uparrow\rangle + \ldots + \alpha_4 |\downarrow\downarrow\rangle \tag{12}$$

such that it has total spin zero, i.e. that all components of $\hat{\boldsymbol{J}}$ annihilate this vector. The total spin operator is

$$\hat{\boldsymbol{J}} = \frac{1}{2}\boldsymbol{\sigma} \otimes \mathbb{1} + \frac{1}{2}\mathbb{1} \otimes \boldsymbol{\sigma}. \tag{13}$$

(iii) Express $|\psi\rangle$ in terms of $|\rightarrow\rangle$ and $|\leftarrow\rangle$ basis. How can you interpret the result?

(iv) Show by a direct calculation that $|\psi\rangle$ cannot be expressed as a product state $|\alpha\rangle \otimes |\beta\rangle$

(v) Calculate the reduced density matrix associated to $|\psi\rangle$ if we can no longer perform any measurements on the second particle (i.e. trace over Hilbert space of the second particle).

(vi) Find the entanglement entropy associated to this reduced density matrix and explain how the result proves that $|\psi\rangle$ was not a product state.

General information

The *lecture* takes place on:
Monday at 10:00 - 12:00 c.t. in B 052 (Theresienstraße 37)
Friday at 10:00 - 12:00 c.t. in B 052 (Theresienstraße 37)
The *central tutorial* takes place on Monday at 12:00 - 14:00 c.t. in B 139 (Theresienstraße 37)
The *webpage* for the lecture and exercises can be found at

`https://www.physik.uni-muenchen.de/lehre/vorlesungen/wise_19_20/T_M1_TV_-Quantum-Mechanics-II`