Prof. Dr. Viatcheslav Mukhanov Winter term 2019/2020

Exercises on Quantum Mechanics II (TM1/TV)

Problem set 3, discussed November 4 - November 8, 2019

Exercise 19

(i) Show that the momentum eigenstates $|\,p\rangle$ are "normalized to the δ-function" if

$$\langle p\,|\,x\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar}px}$$

(ii) Show

$$\hat{x}\,|\,\psi\rangle = \int dx\; x\psi(x)\,|\,x\rangle = i\hbar \int dp \frac{\partial\psi(p)}{\partial p}\,|\,p\rangle$$

and

$$\hat{p}\,|\,\psi\rangle = \int dp\; p\psi(p)\,|\,p\rangle = -i\hbar \int dx \frac{\partial\psi(x)}{\partial x}\,|\,x\rangle$$

where $\psi(x) = \langle x\,|\,\psi\rangle$ and $\psi(p) = \langle p\,|\,\psi\rangle$.

Exercise 20 (central tutorial)

(i) Knowing that $\hat{p}\,|\,p\rangle = p\,|\,p\rangle$ and that $[\hat{q},\hat{p}] = i\hbar$, calculate the matrix element $\langle p_1\,|\,\hat{q}\,|\,p_2\rangle$. You should start by considering the matrix element $\langle p_1\,|\,[\hat{q},\hat{p}]\,|\,p_2\rangle$ and its implications.

Hint: Recall what it has been done in the lecture with $\langle q_1\,|\,\hat{p}\,|\,q_2\rangle$, but this time don't set to zero the constant of integration and keep it.

(ii) How does the previous result change if you replace $\hat{q}$ by $\hat{q} + c(\hat{p})$, where $c(\hat{p})$ is an arbitrary function? What does this mean?

(iii) Consider now a general transformation for the redefinition of position and momentum operator:

$$\hat{Q} = \hat{U}\hat{q}\hat{U}^\dagger \quad , \quad \hat{P} = \hat{U}\hat{p}\hat{U}^\dagger$$

Why is a transformation of the type $\hat{Q} = \hat{U}\hat{q}\hat{V}$ (same for $\hat{P}$) with $\hat{V} \neq \hat{U}^\dagger$ not allowed? What is the condition $\hat{U}$ must satisfy in order to preserve the canonical commutation relation $\left[\hat{Q},\hat{P}\right] = [\hat{q},\hat{p}] = i\hbar$?

(iv) Consider $\hat{U} = e^{\frac{i}{\hbar}\alpha(\hat{p})}$. Working in momentum representation, i.e. $\hat{p} \to p$ and $\hat{q} \to i\hbar\frac{\partial}{\partial p}$ acting on some wave-function $\psi(p)$, find how $\hat{q}$ and $\hat{p}$ are transformed under $\hat{U}$.

Exercise 21

Consider the following Hamilton operator

$$\hat{H} = \frac{\hat{p}^2}{2} + \cosh\hat{x} \tag{1}$$

Write the time independent Schrödinger equation

$$\hat{H}\,|\psi_n\rangle = E_n\,|\psi_n\rangle \tag{2}$$

in components in position and momentum representation.

Exercise 22 (central tutorial)

Let us consider a harmonic oscillator of unit mass, described by the Lagrangian

$$L = \frac{\dot{q}^2}{2} - \frac{\omega^2}{2} q^2 \tag{3}$$

where ω is the constant frequency.

(i) What is the corresponding Hamiltonian? Now promote coordinate and momenta into operators. What is the commutation relation which they should satisfy? Set $\hbar = 1$.

(ii) Let's define creation and annihilation operators respectively as

$$\hat{a}^\dagger = \sqrt{\frac{\omega}{2}} \left(\hat{q} - \frac{i}{\omega} \hat{p} \right) \qquad \hat{a} = \sqrt{\frac{\omega}{2}} \left(\hat{q} + \frac{i}{\omega} \hat{p} \right) \tag{4}$$

Derive the commutation relation which they should satisfy. Express the Hamiltonian in terms of the creation and annihilation operators.

(iii) What is the energy of the ground state? Calculate $\left\langle (\Delta \hat{q})^2 \right\rangle_0 \left\langle (\Delta \hat{p})^2 \right\rangle_0$, where for an operator $\hat{A}$ we have $\left\langle \hat{A} \right\rangle_0 = \langle 0 | \hat{A} | 0 \rangle$, $\hat{a} |0\rangle = 0$ and $\Delta \hat{A} = \sqrt{\left\langle \hat{A}^2 \right\rangle - \left\langle \hat{A} \right\rangle^2}$.

(iv) Now let λ, μ be complex constants. Show the following identities:

$$\begin{aligned} \left[\hat{a}, e^{\lambda \hat{a}^\dagger} \right] &= \lambda e^{\lambda \hat{a}^\dagger} \\ e^{\lambda \hat{a}} e^{\mu \hat{a}^\dagger} &= e^{\mu \lambda} e^{\mu \hat{a}^\dagger} e^{\lambda \hat{a}} \end{aligned} \tag{5}$$

Hint: Consider the function

$$f(\lambda, \mu) = e^{\lambda \hat{a}} e^{\mu \hat{a}^\dagger} e^{-\lambda \hat{a}} e^{-\mu \hat{a}^\dagger} \tag{6}$$

and its derivatives $\frac{\partial f}{\partial \lambda}, \frac{\partial f}{\partial \mu}$.

Exercise 23 (central tutorial)

We define the coherent state as an eigenstate of the annihilation operator:

$$\hat{a} |a\rangle = a |a\rangle \tag{7}$$

(i) Calculate using the results of the previous exercise $\left\langle (\Delta \hat{q})^2 \right\rangle_a \left\langle (\Delta \hat{p})^2 \right\rangle_a$. How can we interpret this result?

(ii) Prove the following relation

$$|a\rangle = e^{a \hat{a}^\dagger} |0\rangle \tag{8}$$

where

$$\hat{a} |0\rangle = 0 \quad \text{and} \quad |a\rangle = \sum_n \frac{a^n}{\sqrt{n!}} |n\rangle \quad \text{with} \quad |n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle \,. \tag{9}$$

(iii) Show that the states $|n\rangle$ are orthonormal eigenvectors of the occupation number operator $\hat{N} = \hat{a}^\dagger \hat{a}$.

(iv) Find $\psi_n(a) = \langle a | n \rangle$ for eigenvectors of $\hat{N}$.

Exercise 24

Consider the Pauli matrices $\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3$.

(i) Show that $\hat{\sigma}_1 \hat{\sigma}_2 = i \hat{\sigma}_3$. What are $\hat{\sigma}_1 \hat{\sigma}_3$ and $\hat{\sigma}_2 \hat{\sigma}_3$?

(ii) Show that $[\hat{\sigma}_1, \hat{\sigma}_2] = 2i\hat{\sigma}_3$ and $\{\hat{\sigma}_1, \hat{\sigma}_2\} = 0$.

The properties above can be generalized and encoded in the following relations:

$$\hat{\sigma}_i\hat{\sigma}_j = \delta_{ij}\mathbb{1} + i\epsilon_{ij}{}^k\hat{\sigma}_k \quad , \quad \{\hat{\sigma}_i, \hat{\sigma}_j\} = 2\delta_{ij} \quad , \quad [\hat{\sigma}_i, \hat{\sigma}_j] = 2i\epsilon_{ij}{}^k\hat{\sigma}_k$$

where $\epsilon_{ij}{}^k$ indicates the Levi-Civita tensor (or *totally anti-symmetric* tensor).

(iii) Let v^j be three components of a (real) vector, define the spin operator as $\hat{\sigma} = (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$ and let

$$v \cdot \hat{\sigma} \equiv v^j\hat{\sigma}_j = v^1\hat{\sigma}_1 + v^2\hat{\sigma}_2 + v^3\hat{\sigma}_3$$

Calculate $(v \cdot \hat{\sigma})^2$ and show that

$$\exp(iv \cdot \hat{\sigma}) = \cos|v| + i\frac{v \cdot \hat{\sigma}}{|v|}\sin|v|$$

where $|v| = \sqrt{v^j v_j}$.

General information

The *lecture* takes place on:
Monday at 10:00 - 12:00 c.t. in B 052 (Theresienstraße 37)
Friday at 10:00 - 12:00 c.t. in B 052 (Theresienstraße 37)
The *central tutorial* takes place on Monday at 12:00 - 14:00 c.t. in B 139 (Theresienstraße 37)
The *webpage* for the lecture and exercises can be found at

`https://www.physik.uni-muenchen.de/lehre/vorlesungen/wise_19_20/T_M1_TV_-Quantum-Mechanics-II`