Prof. Dr. Viatcheslav Mukhanov Winter term 2019/2020

Exercises on Quantum Mechanics II (TM1/TV)

Problem set 2, discussed October 28 - November 1, 2019

Exercise 9 (central tutorial)

(i) Show that if $\hat{A}\hat{B} = \hat{\mathbb{1}} = \hat{C}\hat{A}$ then we have

$$\hat{B} = \hat{A}^{-1} = \hat{C}.$$

Remember that the inverse operator $\hat{A}^{-1}$ satisfies $|\psi\rangle = \hat{A}\,|\chi\rangle$ if and only if $|\chi\rangle = \hat{A}^{-1}\,|\psi\rangle$.

(ii) Given an example of operators $\hat{A}$ and $\hat{B}$ in Hilbert space for which $\hat{A}\hat{B} = \hat{\mathbb{1}}$ holds but for which $\hat{B}\hat{A} \neq \hat{\mathbb{1}}$.

(iii) Let $\hat{A}$ be an operator such that $\hat{A}^2 = \lambda\hat{\mathbb{1}}$ where $\lambda \neq 1$ is a complex number. Write $\left(\hat{A} + \hat{\mathbb{1}}\right)^{-1}$ explicitly in terms of $\hat{A}$.

Exercise 10

Consider the operator $\hat{A} = \frac{d}{dx}$.

(i) Use the Taylor expansion to find out how $e^{\alpha\hat{A}}$ acts on wavefunctions. Interpret the result physically.

(ii) How do operators $\hat{B} \equiv \sinh(\alpha\hat{A})$ and $\hat{C} \equiv \sin(\alpha\hat{A})$ act on wavefunctions?

Exercise 11

Find $\hat{A}^\dagger$ for $\hat{A} = |\varphi\rangle\langle\psi|$. Remember that $\hat{A}^\dagger$ is defined such that for any two vectors $|\psi\rangle$ and $|\chi\rangle$ we have $\left\langle\psi|\hat{A}\chi\right\rangle = \left\langle\hat{A}^\dagger\psi|\chi\right\rangle$.

Exercise 12

(i) Show that for an orthonormal basis $|\delta_j\rangle$ we have the completeness relation

$$\sum_k |\delta_k\rangle\left\langle\delta^k\right| = \hat{\mathbb{1}}.$$

(ii) Consider a product of two operators $\hat{C} = \hat{A}\hat{B}$. Remember that the matrix elements of the operator $\hat{A}$ in the orthonormal basis $|\delta_j\rangle$ were defined as $A^j{}_k \equiv \left\langle\delta^j\right|\hat{A}\,|\delta_k\rangle$. Show that the components of $\hat{C}$ are given by the usual product of matrices.

(iii) Show that the operator $\hat{A}$ can be reconstructed from its components via

$$\hat{A} = \sum_{jk} A^j{}_k\,|\delta_j\rangle\left\langle\delta^k\right|$$

Exercise 13

It was shown in the lecture that the matrix elements of the conjugate operator are

$$\langle \delta^j | \hat{C}^\dagger | \delta_l \rangle = \left[\langle \delta^l | \hat{C} | \delta_j \rangle \right]^* ,$$

i.e. the matrix of components is complex conjugate transpose. Use this to show that $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger \hat{A}^\dagger$.

Exercise 14 (central tutorial)

Consider a change of orthonormal basis $|\delta\rangle \to \left|\tilde{\delta}\right\rangle$ in the Hilbert space described by $U^j{}_k \equiv \left\langle \tilde{\delta}^j | \delta_k \right\rangle$.

(i) Show that $U^j{}_k$ are components of unitary matrix, i.e. $(U^\dagger)^j{}_k = (U^k{}_j)^* = (U^{-1})^j{}_k$.

(ii) Show that the components of bra vectors in the old and in the new basis are related by

$$\tilde{\psi}_k = \sum_j \psi_j (U^\dagger)^j{}_k$$

(iii) Show that the matrix elements of operators transform as

$$\tilde{A}^j_m = \sum_{kl} U^j{}_k A^k{}_l (U^\dagger)^l{}_m .$$

Exercise 15

Consider a Hermitian operator $\hat{A}$ and a unitary operator $\hat{U}$.

(i) Show that the trace of the operator $\hat{A}$ is independent of the choice of the basis. What property of the trace follows from the hermiticity of $\hat{A}$?

(ii) How are spectra of $\hat{A}$ and of $\hat{U}\hat{A}\hat{U}^\dagger$ related?

Exercise 16

Consider a linear operator acting on a Hilbert space such that it maps one orthonormal basis into another one, $\hat{U} |\delta_j\rangle = |\delta'_j\rangle$. How can you write this operator in terms of basis vectors? Find its hermitian conjugate.

Exercise 17 (central tutorial)

The position operator $\hat{x}$ is hermitian. The momenum operator satisfies the commutation relation

$$[\hat{x}, \hat{p}] = i\hbar \hat{\mathbb{1}}.$$

Does this imply that $\hat{p}$ is a hermitian operator? Can there exist finite dimensional matrices $\hat{x}$ and $\hat{p}$ which satisfy these commutation relations?

Exercise 18 (central tutorial)

Check by direct calculation that

$$\int dq'' \left[X^q{}_{q''} P^{q''}{}_{q'} - P^q{}_{q''} X^{q''}{}_{q'} \right] = i\hbar \delta^q_{q'} = i\hbar \delta(q - q')$$

where the matrix elements of $\hat{X}$ are $X^q{}_{q'} \equiv \langle q | \hat{X} | q' \rangle$ and similarly for $\hat{P}$.