Prof. Dr. Viatcheslav Mukhanov Winter term 2019/2020

Exercises on Quantum Mechanics II (TM1/TV)

Problem set 1, discussed October 21 - October 25, 2019

Exercise 1

Recall that in a vector space $\mathcal{V}$ the following axioms have to be fulfilled:
Let $|\psi\rangle, |\phi\rangle \in \mathcal{V}$ and $a, b \in \mathbb{C}$, then:
1) $a(|\psi\rangle + |\phi\rangle) = a|\psi\rangle + a|\phi\rangle$ and $(a+b)|\psi\rangle = a|\psi\rangle + b|\psi\rangle$
2) $a(b|\psi\rangle) = b(a|\psi\rangle)$
3) $\exists \mathbf{0} \in \mathcal{V}$ such that $\forall |\psi\rangle \in \mathcal{V}$, $0 \cdot |\psi\rangle = \mathbf{0}$
4) $1 \cdot |\psi\rangle = |\psi\rangle$
Show that:

(i) $|\psi\rangle + \mathbf{0} = |\psi\rangle$

(ii) $a(|\phi\rangle - |\psi\rangle) = a|\phi\rangle - a|\psi\rangle$

Exercise 2

Let $\mathcal{V}$ be a $\mathbb{C}$ vector space.

(i) What are the necessary conditions on all $|\psi\rangle \in \mathcal{V}$ for $\mathcal{V}$ to be a Hilbert space?

(ii) Now let $|\psi\rangle$ be an ordered sequence of complex numbers ψ_i, where $\sum_i |\psi_i|^2 < \infty$. Is it possible to define a scalar product $(\bullet, \bullet)$ according to $(|\psi\rangle, |\phi\rangle) = \sum_i \psi_i^* \phi_i$?

(iii) Is it possible to define a hermitian scalar product $(\bullet, \bullet)$ according to $(|\psi\rangle, |\phi\rangle) = \sum_i \psi_i \phi_i$?

Exercise 3 (central tutorial)

Take the set $L^2(a, b)$ of all complex square integrable functions on the interval $[a, b]$ and the canonical addition map and scalar product as:
1) $\psi, \phi \in L^2(a, b)$: $(\psi + \phi)(x) := \psi(x) + \phi(x)$
2) $(\psi, \phi) := \int_a^b \psi^*(x)\phi(x)\mathrm{d}x$
Show that the emerging space $\mathcal{T}_F$ is a Hilbert space.

Exercise 4 (central tutorial)

Let $\mathcal{H}$ and $\mathcal{H}^*$ be Hilbert spaces dual to each other. A bra vector $\langle\phi| \in \mathcal{H}^*$ corresponding to a ket vector $|\phi\rangle \in \mathcal{H}$ is defined via the scalar product as $\langle\phi|\chi\rangle \equiv (|\phi\rangle, |\chi\rangle)$.
Take $|\Psi\rangle = a|\phi\rangle + b|\psi\rangle \in \mathcal{H}$ and $\langle\Phi| = a^*\langle\phi| + b^*\langle\psi| \in \mathcal{H}^*$. Show that $\langle\Phi|$ corresponds to $|\Psi\rangle$, i.e. $\langle\Phi| = \langle\Psi|$.

Exercise 5

Let $\mathcal{H}$ be a Hilbert space with respect to the inner product $\langle\bullet|\bullet\rangle$. Prove the Cauchy-Schwartz inequality $|\langle\psi|\phi\rangle| \leq \||\psi\rangle\| \, \||\phi\rangle\|$, where $\||\psi\rangle\| = \sqrt{\langle\psi|\psi\rangle}$.

Exercise 6 (central tutorial)

The set $\{|\phi_i\rangle\}$ forms a basis of the Hilbert space $\mathcal{H}$. Prove the sequence

$$|\chi_1\rangle = \frac{|\phi_1\rangle}{\||\phi_1\rangle\|}, \quad |\chi_k\rangle = \frac{|\phi_k\rangle - \sum_{j=1}^{k-1} \langle\chi_j|\phi_k\rangle\,|\chi_j\rangle}{\left\||\phi_k\rangle - \sum_{j=1}^{k-1} \langle\chi_j|\phi_k\rangle\,|\chi_j\rangle\right\|} \quad \forall k > 1$$

forms an orthonormal basis of $\mathcal{H}$.

Exercise 7

Let $\mathcal{T}_F$ be the Hilbert space of square integrable complex functions defined in Exercise 3.

(i) Which of the following operators are linear operators?

$$\hat{A} = (\bullet)^2, \hat{B} = \frac{\mathrm{d}}{\mathrm{d}x}(\bullet), \hat{C} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}(\bullet), \hat{D} = g(x)(\bullet), \hat{E} = g(x)(\bullet)^3$$

(ii) Which of the following symbols can be interpreted as operators on $\mathcal{T}_F$?

a) $\int \psi(x)(\bullet)\mathrm{d}x$

b) $\int (\bullet)k(x,y)(*)\mathrm{d}x\mathrm{d}y$

c) $\int k(x,y)(\bullet)\mathrm{d}x$

Exercise 8

Let $\hat{A}, \hat{B}, \hat{C}$ be linear operators acting on a Hilbert space. Prove

$$\hat{A}(\hat{B} + \hat{C}) = \hat{A}\hat{B} + \hat{A}\hat{C}$$

General information

The *lecture* takes place on:
Monday at 10:00 - 12:00 c.t. in B 052 (Theresienstraße 37)
Friday at 10:00 - 12:00 c.t. in B 052 (Theresienstraße 37)
The *central tutorial* takes place on Monday at 12:00 - 14:00 c.t. in B 139 (Theresienstraße 37)
The *webpage* for the lecture and exercises can be found at

`https://www.physik.uni-muenchen.de/lehre/vorlesungen/wise_19_20/T_M1_TV_-Quantum-Mechanics-II`