Prof. Dr. Viatcheslav Mukhanov Winter term 2019/2020

Exercises on Quantum Mechanics II (TM1/TV)

Problem set 13, discussed January 27 - January 31, 2020

EPR paradox vs Bell's inequality - Central Tutorial

Exercise 72 - Central Tutorial

One way to understand the ability of quantum mechanics to protect from information detection is from the fact that non-orthogonal states cannot be perfectly distinguished.

(i) In the course of a quantum key distribution protocol, suppose that Alice randomly chooses one of the following two states and transmits it to Bob:

$$|\,\phi_0\rangle = \frac{1}{\sqrt{2}}(|\,0\rangle + |\,1\rangle), \qquad \text{or} \qquad |\,\phi_1\rangle = \frac{1}{\sqrt{2}}(|\,0\rangle + i\,|\,1\rangle). \tag{1}$$

Eve intercepts the qubit and performs a measurement to identify the state. The measurement consists of the orthogonal states $|\,\psi_0\rangle$ and $|\,\psi_1\rangle$, and Eve guesses the transmitted state was $|\,\phi_0\rangle$ when she obtains the outcome $|\,\psi_0\rangle$, and so forth. What is the probability that Eve correctly guesses the state, averaged over Alice's choice of the state for a given measurement? What is the optimal measurement Eve should make, and what is the resulting optimal guessing probability?

(ii) Now suppose Alice randomly chooses between two states $|\,\phi_0\rangle$ and $|\,\phi_1\rangle$ separated by an angle θ on the Bloch sphere. What is the measurement which optimizes the guessing probability? What is the resulting probability of correctly identifying the state, expressed in terms of θ? In terms of the states? *Hint:* Review the main results from exercise 60 and think about how to represent the measurement probability $P = |\langle a\,|\,b\rangle|^2$ in terms of Block sphere vectors.

Solution

(i) The the probability of correctly guessing, averaged over Alice's choice of the state is

$$P_{guess} = \frac{1}{2}\left(|\langle\psi_0\,|\,\phi_0\rangle|^2 + |\langle\psi_1\,|\,\phi_1\rangle|^2\right). \tag{2}$$

To optimize the choice of measurement, suppose $|\,\psi_0\rangle = a\,|\,0\rangle + b\,|\,1\rangle$ for some $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$. Then $|\,\psi_1\rangle = -b^*\,|\,0\rangle + a^*\,|\,1\rangle$ is orthogonal as intended. Using this in eq. 2,

$$P_{guess} = \frac{1}{2}\left(\left|\frac{1}{\sqrt{2}}(a^* + b^*)\right|^2 + \left|\frac{1}{\sqrt{2}}(-b + ia)\right|^2\right) = \frac{1}{2}\left(1 + 2\mathrm{Re}\left[\left(\frac{1-i}{2}\right)ab^*\right]\right). \tag{3}$$

If we express $a = \alpha e^{i\theta}$ and $b = \beta e^{i\eta}$ for $\alpha, \beta, \theta, \eta \in \mathbb{R}$, then we get

$$P_{guess} = \frac{1}{2}\left(1 + 2\alpha\beta\mathrm{Re}\left[\left(\frac{1-i}{2}\right)e^{i(\theta-\eta)}\right]\right). \tag{4}$$

To maximize, we ought to choose $\alpha = \beta = 1/\sqrt{2}$ and we may set $\eta = 0$ since only the difference $\theta - \eta$ is relevant. Now we have

$$P_{guess} = \frac{1}{2}\left(1 + \mathrm{Re}\left[\left(\frac{1-i}{2}\right)e^{i\theta}\right]\right) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\mathrm{Re}\left[e^{-i\pi/4}e^{i\theta}\right]\right). \tag{5}$$

from which it is clear that the best thing to do is to set $\theta = \pi/4$ to get

$$P_{guess} = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) \approx 85.4\%. \tag{6}$$

The basis states making up the measurement are

$$\begin{aligned} |\,\psi_0\rangle &= \frac{1}{\sqrt{2}}\left(e^{i\pi/4}\,|\,0\rangle + |\,1\rangle\right) \\ |\,\psi_1\rangle &= \frac{1}{\sqrt{2}}\left(-\,|\,0\rangle + e^{-i\pi/4}\,|\,1\rangle\right). \end{aligned} \tag{7}$$

(ii) *Probability of measurement in terms of Bloch sphere vectors:* Recall from exercise 60 that a general state of any two state system can be represented as a vector on the Bloch sphere, $|\,\boldsymbol{r}\rangle \leftrightarrow \boldsymbol{r}$. One can then express the state on the Bloch sphere as a density matrix,

$$\rho_{\boldsymbol{r}} = \frac{1}{2}(1 + \boldsymbol{r}\cdot\boldsymbol{\sigma}). \tag{8}$$

The probability of measuring a system in a state $|\,\boldsymbol{m}\rangle \leftrightarrow \boldsymbol{m}$ then is

$$P_{\boldsymbol{r}}(\boldsymbol{m}) = |\langle \boldsymbol{m}\,|\,\boldsymbol{r}\rangle|^2 = \langle \boldsymbol{m}\,|\,\boldsymbol{r}\rangle\,\langle \boldsymbol{r}\,|\,\boldsymbol{m}\rangle = \langle \boldsymbol{m}\,|\,\hat{\rho}_{\boldsymbol{r}}\,|\,\boldsymbol{m}\rangle = \mathrm{tr}(|\,\boldsymbol{m}\rangle\,\langle \boldsymbol{m}\,|\,\hat{\rho}_{\boldsymbol{r}}) = \mathrm{tr}(\hat{\rho}_{\boldsymbol{m}}\hat{\rho}_{\boldsymbol{r}}). \tag{9}$$

Now using eq. 8 and the identities $\mathrm{tr}(\sigma^i) = 0$, $\mathrm{tr}(\sigma^i\sigma^j) = 2\delta^{ij}$, one can easily verify

$$P_{\boldsymbol{r}}(\boldsymbol{m}) = \frac{1}{2}(1 + \boldsymbol{r}\cdot\boldsymbol{m}). \tag{10}$$

The point of this exercise is to show that thinking in terms of the Bloch sphere is a lot more intuitive than just taking a brute force approach as we did in the solution of the previous exercise. Let $\boldsymbol{n}_0$ and $\boldsymbol{n}_1$ be the Bloch vectors of the two states. Call $\boldsymbol{m}$ the Bloch vector associated with one of the two basis vectors of the measurement, specifically the one which indicates that the state is $|\,\phi_0\rangle$ (the other is associated with $-\hat{m}$). The guessing probability takes the form

$$\begin{aligned} P_{guess} &= \frac{1}{2}\left(|\langle\psi_0\,|\,\phi_0\rangle|^2 + |\langle\psi_1\,|\,\phi_1\rangle|^2\right) \\ &= \frac{1}{2}\left(\frac{1}{2}\left(1 + \boldsymbol{n}_0\cdot\boldsymbol{m}\right) + \frac{1}{2}\left(1 - \boldsymbol{n}_1\cdot\boldsymbol{m}\right)\right) \\ &= \frac{1}{4}\left(2 + \boldsymbol{m}\cdot(\boldsymbol{n}_0 - \boldsymbol{n}_1)\right). \end{aligned} \tag{11}$$

The optimal $\hat{m}$ lies along $\boldsymbol{n}_0 - \boldsymbol{n}_1$ and has unit length, i.e

$$\boldsymbol{m} = \frac{\boldsymbol{n}_0 - \boldsymbol{n}_1}{|\boldsymbol{n}_0 - \boldsymbol{n}_1|} = \frac{\boldsymbol{n}_0 - \boldsymbol{n}_1}{\sqrt{2 - 2\cos\theta}} \tag{12}$$

Therefore,

$$P_{guess} = \frac{1}{4}\left(2 + \sqrt{2 - 2\cos\theta}\right) = \frac{1}{2}\left(1 + \sin\frac{\theta}{2}\right). \tag{13}$$

Finally, we should check that this gives sensible results. When $\theta = 0$, $P_{guess} = 1/2$, as it should. On the other hand, the states $|\,\phi_k\rangle$ are orthogonal for $\theta = \pi$, and indeed $P_{guess} = 1$ in this case. In the previous exercise we investigated the $\theta = \pi/2$ and here we immediately find $P_{guess} = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)$, as before.

Exercise 73

Consider a system consisting of two coils as in the lecture, a superconducting coil with induction I and current $J = \dot{q}$ interacting with a second coil with current $j(t)$ flowing in it. The latter is free to rotate, where the angle about the rotation axis is denoted by φ and has moment of inertia T. The Lagrangian of the system is given by

$$L = \frac{1}{2}I\dot{q}^2 + \frac{1}{2}T\dot{\varphi}^2 - M_0\varphi\dot{q}j(t), \tag{14}$$

where in the interaction term, the mutual inductance is approximated by $M(\varphi) \approx M_0\varphi$, with $M_0 = const.$ Verify that the Hamiltonian is given by

$$H = \frac{p_q^2}{2I} + \frac{p_\varphi^2}{2T} + \frac{Ig^2(t)\varphi^2}{2} + g(t)\varphi p_q \ , \tag{15}$$

where $g(t) = \frac{M_0}{I} j(t)$. Check that the condition for a non-demolition measurement is satisfied.

Solution First of all we obtain the conjugated momenta:

$$p_q = \frac{\partial L}{\partial \dot{q}} = I\dot{q} - M_0\varphi j(t) \ , \qquad p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = T\dot{\varphi} \tag{16}$$

Inverting these relations to obtain $\dot{q}$ and $\dot{\varphi}$ in terms of the momenta we can plug them in $H = p_q\dot{q} + p_\varphi\dot{\varphi} - L$. After the trivial computation we obtain the desired result.
It is easily seen, that $[\hat{p}_q, H] = 0$, and before and after the measurement $p_q = I\dot{q}$ and hence the state of the system is unchanged after the measurement.

Exercise 74

Let p_q in Exercise 73 be given by $p_q = -\frac{p_\varphi(t_f) - p_\varphi(t_i)}{g_0 \triangle t} - Ig_0\varphi(t_i)$. Why is the uncertainty $\triangle p_q$ bounded by

$$\triangle p_q \geq \sqrt{\frac{\triangle p_\varphi^2}{g_0^2 \triangle t^2} + I^2 g_0^2 \triangle \varphi_i^2} \quad ? \tag{17}$$

Solution Neglecting correlations or assuming independent variables yields the common formula among experimental physicists to calculate error propagation, the variance formula:

$$\triangle p_q = \sqrt{\left(\frac{\partial p_q}{\partial p_\varphi}\right)^2 \triangle p_\varphi^2 + \left(\frac{\partial p_q}{\partial \varphi}\right)^2 \triangle \varphi^2} \tag{18}$$

(It is important to note that this formula is a good estimation for the standard deviation as long as the involved deviations are small enough, otherwise it receives corrections and becomes larger justifying the $\geq$.)

Therefore we straightforwardly obtain (17).

Exercise 75

Let $\hat{\tau} = m\frac{\hat{x}}{\hat{p}} \equiv \frac{m}{2}(\hat{x}\hat{p}^{-1} + \hat{p}^{-1}\hat{x})$ be the "time operator" of a quantum clock. Check that the uncertainty relation $[\hat{\tau}, \hat{E}] = i\hbar$ is valid, where the "energy operator" $\hat{E}$ is given by $\hat{p}^2/2m$. Shortly interpret the result compared to the violation of the energy-time uncertainty principle in the lecture.

Solution

$$[\hat{\tau}, \hat{E}] = \hat{\tau}\hat{E} - \hat{E}\hat{\tau} = \frac{1}{4}(\hat{x}\hat{p} + \hat{p}^{-1}\hat{x}\hat{p}^2 - \hat{p}^2\hat{x}\hat{p}^{-1} - \hat{p}\hat{x}) = \frac{1}{4}(\hat{x}\hat{p} + \hat{p}^{-1}(\hat{x}\hat{p})\hat{p} - \hat{p}(\hat{p}\hat{x})\hat{p}^{-1} - \hat{p}\hat{x}) \tag{19}$$

with $\hat{x}\hat{p} = \hat{p}\hat{x} + i\hbar$ and $\hat{p}\hat{x} = \hat{x}\hat{p} - i\hbar$ follows

$$[\hat{\tau}, \hat{E}] = i\hbar \tag{20}$$

We observe that for a proper/inertial/observable time the uncertainty principle holds while for the external times (such that measurements are carried out with clocks that are not dynamically connected with the objects studied in the experiment) used in the lecture it doesn't. For the interested reader more information can be found in section 3.2 of `https://arxiv.org/pdf/quant-ph/0105049.pdf`.

Solution

General information

The *lecture* takes place on:
Monday at 10:00 - 12:00 c.t. in B 052 (Theresienstraße 37)
Friday at 10:00 - 12:00 c.t. in B 052 (Theresienstraße 37)
The *central tutorial* takes place on Monday at 12:00 - 14:00 c.t. in B 139 (Theresienstraße 37)
The *webpage* for the lecture and exercises can be found at

`https://www.physik.uni-muenchen.de/lehre/vorlesungen/wise_19_20/T_M1_TV_-Quantum-Mechanics-II`