Ludwig-Maximilians-Universität München
Department of Physics

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T_M1/TV: Quantum Mechanics II

Final Exam

20 February 2017

- Please make sure that each sheet of paper carries your name and/or ID (Matrikelnummer).
- Please indicate very clearly which problem any of your calculations/results belong to, and start a new page for each problem.
- The maximal number of points is 70.
  It is not necessary to score the top mark in order to receive the best grade.
- Please answer the questions in English or German.

I agree that my result will be published together with my ID on the lecture webpage. Signature: ______________________________

Name: ______________________________

ID: ______________________________

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Problem 1: Short questions [14 points]

a) The classical Lagrangian of a particle in an electromagnetic vector potential $\vec{A}$ is

$$L = \tfrac{1}{2}m\dot{\vec{x}}^{\,2} + \tfrac{e}{c}\vec{A}\cdot\dot{\vec{x}}\,.$$

The corresponding Hamiltonian is naturally a function of the phase space variables $\vec{x}$ and $\vec{p}$. Calculate the Hamiltonian in terms of the configuration variables $\vec{x}$ and $\dot{\vec{x}}$ instead.

b) Consider a system of two particles of spin $\frac{1}{2}$.

(i) Is the state $|\psi\rangle = \frac{1}{\sqrt{2}}\big(|+-\rangle - |++\rangle\big)$ an entangled state? Please explain your answer.

(ii) Assume the particles are identical. Why can a wave function of the system with $|\psi\rangle$ as its spin part not exist?

c) A parameter of a Hamiltonian with non-degenerate discrete spectrum which is bounded from below is changed adiabatically. The spectrum remains discrete for all values of the parameter.

(i) If the system was initially in the energy eigenstate $|n\rangle$, and after the deformation is in the state $|m\rangle$, how is m related to n, and why?

(ii) With the information you have, what can you say about the energy of the state $|m\rangle$, compared with the energy of the state $|n\rangle$

α) if the deformation describes a closed loop in parameter space?

β) if the value of the parameter at the end point is different from the value at the initial point?

d) Consider scattering of individual photons on electrons bound to an atom. Suppose the energy of the incoming photon is $\hbar\omega$, and the energy of the outging photon is $\hbar\omega'$. Consider

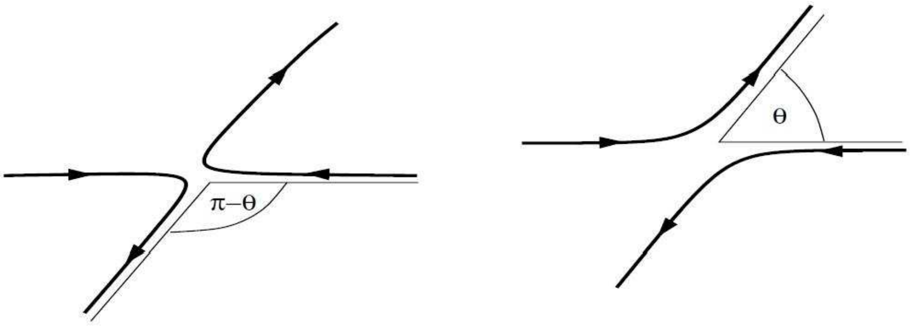
(i) Anti-Stokes scattering,

(ii) Rayleigh scattering.

For each case, indicate if $\hbar\omega'$ is larger, smaller, or equal to $\hbar\omega$.

Problem 1: Short questions (continued)

e) In an elastic scattering process of two interacting bosonic particles of the same mass, consider the following trajectories in the center-of-mass frame:



For distinguishable particles, let the two trajectories correspond to scattering amplitudes $f(\pi-\theta)$ and $f(\theta)$, respectively. Suppose your experiment does not distinguish between the particles, and only measures if any particle has been scattered at all in any particular direction. Suppose the differential cross section is

(i) $\frac{d\sigma}{d\Omega} = |f(\theta)|^2 + |f(\pi-\theta)|^2$,

(ii) $\frac{d\sigma}{d\Omega} = |f(\theta) + f(\pi-\theta)|^2$.

Which case do you expect if the particles are identical, and why?

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Problem 2: Entangled states [9 points]

Consider three Hilbert spaces $\mathcal{H}_i$ with $\dim\mathcal{H}_i = 3$, $i = 1, 2, 3$. Take the tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$. Let $|0\rangle, |1\rangle, |2\rangle$ be an orthonormal basis in a single three-dimensional Hilbert space. Then a basis of the tensor product is given by $|xyz\rangle$, where $x, y, z \in \{0, 1, 2\}$. Pick the following two states in $\mathcal{H}$:

(i) $|\psi\rangle = |000\rangle$

(ii) $|\phi\rangle = \frac{1}{\sqrt{3}} (|012\rangle + |120\rangle + |201\rangle)$

We consider the subsystem consisting of the first two particles, with Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

a) Compute the reduced density matrix for the cases (i) and (ii).

b) Compute the entanglement entropy for the cases (i) and (ii), using your result from part a).

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Problem 3: Time-dependent perturbations of the Harmonic Oscillator [13 points]

Consider a one-dimensional Harmonic Oscillator of mass m and frequency ω with Hamiltonian

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \, .$$

For the time interval $t \in [0, T]$ let the system be perturbed by the additional potential $\lambda V(x, t)$, where $V(x, t)$ is a time-dependent operator in the Schrödinger picture, and λ is a small real parameter.

For $t < 0$ let the system be in the eigenstate $|m\rangle$ of H_0. Let $|\tilde{\psi}_m(t)\rangle_I$ denote the state of the system and $\lambda\tilde{V}_I(x, t)$ denote the perturbation at time t in the interaction picture.

a) Show that
$$i\hbar\partial_t|\tilde{\psi}_m(t)\rangle_I = \lambda\tilde{V}_I(x, t)|\tilde{\psi}_m(t)\rangle_I \, .$$

b) For $t \in [0, T]$, find $|\tilde{\psi}_m(t)\rangle_I$ to second order in λ.

c) Suppose $V(x, t) = f(t)\, x^N$ for $N \in \mathbb{N}$, with $f(t)$ a real-valued function of t. We are interested in selection rules for transition probabilities $P_{m\to n}(T)$ between eigentstates $|m\rangle$ and $|n\rangle$ of H_0 under this perturbation. Fixing N and m, show which $P_{m\to n}(T)$ are possibly non-zero in *first* order perturbation theory.

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Problem 4: Elastic scattering in various potentials [10 points]

a) Consider elastic scattering in a dipole potential

$$V(\vec{r}) = -\frac{e^2}{|\vec{r}+\vec{a}|} + \frac{e^2}{|\vec{r}-\vec{a}|}$$

for electric charge e and constant vector $\vec{a}$. Does this potential fall off rapidly enough for an application of the first Born approximation? Please explain your answer.

b) Consider elastic scattering in the Gaussian well potential

$$V(\vec{r}) = V_0 e^{-\frac{\vec{r}^2}{r_0^2}}$$

for $r_0 > 0$. Compute the scattering amplitude in the first Born approximation.

c) Consider a scattering amplitude

$$f(\vec{k}, \vec{k}') = A\, e^{-b(\vec{k}-\vec{k}')^2} \, .$$

Here $\vec{k}$ is the wave vector of the incident and $\vec{k}'$ the wave vector of the scattered wave, and A and b are dimensionful constants. Calculate the differential cross section $\frac{d\sigma}{d\Omega}$, and the total cross section σ. What is the value of σ in the limit $b\vec{k}^2 \to 0$?

Problem 5: Fermionic Pair Distribution [13 points]

Consider N non-interacting free fermions in a one-dimensional system of length L, which are in the ground state

$$|\phi_0\rangle = |n_{k_1\sigma_1},\ n_{k_2\sigma_2},\dots\rangle\,,$$

where $n_{k_i\sigma_i} = 1$ if $|k_i| \leq k_F$, and $n_{k_i\sigma_i} = 0$ if $|k_i| > k_F$.

a) Consider the fermionic field operator of specific spin σ,

$$\hat{\Psi}_\sigma(x) = \frac{1}{\sqrt{L}} \sum_{k_i} e^{ik_i\cdot x} a_{k_i\sigma}\,.$$

Show that the expectation value of the particle density is given by

$$\langle\phi_0|\hat{\Psi}^\dagger_\sigma(x)\hat{\Psi}_\sigma(x)|\phi_0\rangle = \frac{N_\sigma}{L}\,,$$

where $N_\sigma = \sum_{k_i} n_{k_i\sigma}$.

b) Show that

$$\langle\phi_0|\hat{a}^\dagger_{k\sigma}\hat{a}^\dagger_{q\sigma}\hat{a}_{q'\sigma}\hat{a}_{k'\sigma}|\phi_0\rangle = (\delta_{kk'}\delta_{qq'} - \delta_{kq'}\delta_{qk'})n_{k\sigma}n_{q\sigma}.$$

c) Define the same-spin pair distribution function

$$g_{\sigma\sigma}(x-x') = \left(\frac{L}{N_\sigma}\right)^2 \langle\phi_0|\hat{\Psi}^\dagger_\sigma(x)\hat{\Psi}^\dagger_\sigma(x')\hat{\Psi}_\sigma(x')\hat{\Psi}_\sigma(x)|\phi_0\rangle\,.$$

Using the result from part b) show that

$$g_{\sigma\sigma}(x-x') = 1 - \frac{1}{N_\sigma^2}\left|\sum_{k_i} n_{k_i\sigma} e^{i(x-x')k_i}\right|^2\,.$$

d) Obviously we have $g_{\sigma\sigma}(x-x') \leq 1$. Show that $g_{\sigma\sigma}(x-x') \geq 0$.

e) The pair distribution function for *different* spins $\sigma \neq \sigma'$ is

$$g_{\sigma\sigma'}(x-x') := \frac{L^2}{N_\sigma N_{\sigma'}} \langle\phi_0|\hat{\Psi}^\dagger_\sigma(x)\hat{\Psi}^\dagger_{\sigma'}(x')\hat{\Psi}_{\sigma'}(x')\hat{\Psi}_\sigma(x)|\phi_0\rangle = 1\,.$$

Compare with the result for $g_{\sigma\sigma}$, and explain the difference.

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Problem 6: Klein-Gordon equation and electromagnetic field [11 points]

The Klein-Gordon equation for a scalar quantum field ψ coupled to a classical electromagnetic field is given by

$$\left(D_\mu D^\mu + \frac{m^2c^2}{\hbar^2}\right)\psi = 0\,,$$

where the covariant derivative is

$$D_\mu = \partial_\mu + \frac{ie}{\hbar c}A_\mu\,.$$

Under gauge transformations

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu\lambda, \quad \psi \to \psi' = e^{\frac{ie\lambda}{\hbar c}}\psi\,,$$

where λ is a function.

a) Show that $D'_\mu\psi' = (D_\mu\psi)'$.

b) Show that the current

$$j^\mu = \frac{i\hbar}{2m}\left(\psi^\dagger D^\mu\psi - \psi\bar{D}^\mu\psi^\dagger\right)$$

is gauge invariant.

c) Show that the current j^μ is conserved.

d) Spell out the Klein Gordon equation for a constant magnetic field.