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Problem Set 8: Bosonization

In one dimensional systems, there is an exact way of transforming a fermionic theory into a bosonic one: bosonization. In this problem set, we go through a step by step derivation of the bosonized theory. The derivation is based on the reference Annalen Phys. 7 (1998) by J. von Delft and H. Schoeller. We will use the following assumptions:

- The system is one-dimensional. We consider the Hamiltonian of the Tomonaga–Luttinger model ($\hbar v_F = 1$, $:\cdots:$ is defined in Ex. 1),

$$H_0 = \tfrac{2\pi}{L} \sum_{k\eta} \left(m_k - \tfrac{1}{2}\right) : c^\dagger_{k\eta} c_{k\eta} : . \tag{1}$$

- The theory can be formulated in terms of fermionic creation and annihilation operators, $c^\dagger_{k\eta}$ and $c_{k\eta}$, which satisfy $\{c_{k\eta}, c^\dagger_{k'\eta'}\} = \delta_{kk'}\delta_{\eta\eta'}$, where η labels different species, e.g. spins or left/right-movers.

- The momentum k is quantized such that $k = \frac{2\pi}{L}\left(m_k - \frac{1}{2}\right)$ with $m_k \in \mathbb{Z}$, and L a length (system size). Note that k is unbounded.

Exercise 1. *Groundwork in terms of fermions*

(a) Introduce a field ψ_η via the Fourier transform

$$\psi_\eta(x) = \sqrt{\tfrac{2\pi}{L}} \sum_k e^{-ikx} c_{k\eta}. \tag{2}$$

Show that this field satisfies the canonical anticommutation relations (for fields with periodicity L)

$$\{\psi_\eta(x), \psi^\dagger_{\eta'}(x')\} = \delta_{\eta\eta'}\, 2\pi \sum_{n\in\mathbb{Z}} \delta(x - x' - nL)\, e^{i\pi n}, \tag{3}$$

$$\{\psi_\eta(x), \psi_{\eta'}(x')\} = 0 = \{\psi^\dagger_\eta(x), \psi^\dagger_{\eta'}(x')\}. \tag{4}$$

(b) Bosonization can be understood as an operator identity on Fock space. To show this, let us look at the structure of the Fock space of the fermions $c_{k\eta}$. We therefore define the vacuum (or Fermi sea) $|0\rangle_0$ as usual via

$$c_k|0\rangle_0 = 0 \quad \text{for} \quad k > 0, \qquad c^\dagger_k|0\rangle_0 = 0 \quad \text{for} \quad k < 0. \tag{5}$$

To deal with the infinite number of states below the Fermi surface (and their resulting infinite energy), we introduce normal-ordering $:\cdots:$ with respect to this vacuum, defined as moving all operators annihilating the vacuum to the right. It can be rewritten as $:ABC\cdots: = ABC\cdots - {}_0\langle 0|ABC\ldots|0\rangle_0$.

We further introduce the particle number operator as

$$N_\eta = \sum_k : c^\dagger_{k\eta} c_{k\eta} : . \tag{6}$$

We will show that, for each N_η, fluctuations inside of the corresponding space are described by adding electron-hole pairs, which can be described by a bosonic field.

Show that the bosonic operators

$$b^\dagger_{q\eta} = \tfrac{i}{\sqrt{m_q}} \sum_k c^\dagger_{k+q\eta} c_{k\eta}, \qquad b_{q\eta} = \tfrac{-i}{\sqrt{m_q}} \sum_k c^\dagger_{k-q\eta} c_{k\eta}, \tag{7}$$

with $q = \frac{2\pi}{L} m_q > 0$, $m_q \in \mathbb{N}$, obey a bosonic algebra. Be careful when subtracting two terms with infinite contributions from $k < 0$.

(c) Let's now focus on the vacuum of the b operators. Denote by $|\vec{N}\rangle_0$ the state with eigenvalues $\vec{N} = (N_1, \dots N_\eta, \dots)$, which does *not* contain any electron-hole pairs. Visualize this state and the action of $b^{(\dagger)}_{q\eta}$ for one species η, and convince yourself that

$$b_{q\eta} |\vec{N}\rangle_0 = 0, \tag{8}$$

for all q and η. For the following, note that the normal ordering prescription of the bosons is basically determined by the normal ordering prescription of the fermions.

Exercise 2. *Completeness of the bosonic representation*

Let us check that we do not any lose information when going from the fermions to the bosons. For simplicity, you may restrict yourself to **only one species η in this exercise**. The Fock space $\mathcal{F}_c = \oplus_N \mathcal{H}_N$, which is the direct sum of Hilbert spaces with fixed particle number N, is spanned by arbitrary combinations of the fermion operators $c^{(\dagger)}_k$ acting on the vacuum state $|N\rangle_0$. We will show that it is identical to the Fock space $\mathcal{F}_b$, spanned by arbitrary combinations of the bosonic operators $b^{(\dagger)}_q$ acting on the set of all N-particle states $\{|N\rangle_0, N \in \mathbb{Z}\}$.

(a) Show that $\mathcal{F}_b \subseteq \mathcal{F}_c$.

(b) To further prove $\mathcal{F}_b = \mathcal{F}_c$, we compare the grand-canonical partition functions of both spaces. These are sums over positive definite quantities. If there is some state inside $\mathcal{F}_c$ which is not inside of $\mathcal{F}_b$, then the partition functions must disagree. So, as a first step, compute the grand-canonical partition function for $\mathcal{F}_c$.

(c) We now derive a bosonic representation of the Hamiltonian H_0. Show that ${}_0\langle \vec{N}|H_0|\vec{N}\rangle_0 = \frac{\pi}{L} N^2$ and $[H_0, b^\dagger_q] = q b^\dagger_q$. Then, the Hamiltonian is given by

$$H_0 = \sum_{q>0} q\, b^\dagger_q b_q + \tfrac{\pi}{L} N^2. \tag{9}$$

(d) Find an expression for the grand-canonical partition function $\mathcal{F}_b$.

Hint: $\mathcal{F}_b$ is spanned by states of the form $|N; \{n_q\}\rangle = \prod_{q>0} \frac{b_q^{\dagger n_q}}{(n_q!)^{1/2}} |N\rangle_0$.

(e) Show that the partition functions in the fermionic and bosonic representation are equal.

Hint: Use the following two identities

$$\sum_{n=-\infty}^{\infty} y^{n^2} x^{2n} = \prod_{m=1}^{\infty} (1 - y^{2m})(1 + y^{2m-1}x^2)(1 + y^{2m-1}x^{-2}), \tag{10}$$

$$\sum_{M=0}^{\infty} P(M) y^M = \prod_{n=1}^{\infty} \frac{1}{1 - y^n}, \tag{11}$$

where $x \neq 0$ and $|y| < 1$. $P(M)$ describes the number of partitions of M.

Exercise 3. *Bosonization*

To prove the bosonization identity, we need some additional ingredients:

- The bosonic operators $b_{q\eta}^{(\dagger)}$ act within Hilbert spaces of fixed particle number. To mediate between those, we introduce the so-called Klein factors, which transform $|\vec{N}\rangle$ to $|\vec{N}'\rangle$, changing the particle number, as follows

$$F_\eta = \sum_{\vec{N}} \sum_{n_q} |N_1, \ldots, N_\eta - 1, \ldots, N_M; \{n_q\}\rangle\langle N_1, \ldots, N_\eta, \ldots, N_M; \{n_q\}|\, \hat{T}_\eta, \tag{12}$$

  where $\hat{T}_\eta = (-1)^{\sum_{\mu=1}^{\eta-1} \hat{N}_\mu}$ counts the phase. It is easy to check that these operators satisfy $F_\eta^\dagger F_\eta = F_\eta F_\eta^\dagger = 1$ and are fermionic.

- We introduce a bosonic field via

$$\varphi_\eta(x) = -\sum_{q>0} \frac{1}{\sqrt{m_q}} e^{-iq(x - ia/2)} b_{q\eta} \tag{13}$$

  (and its hermitian conjugate), where a is a positive regularizer (the lattice constant).

(a) Show that $:\psi_\eta^\dagger(x)\psi_\eta(x): = \partial_x\big(\varphi_\eta(x) + \varphi_\eta^\dagger(x)\big) + \frac{2\pi}{L}\hat{N}_\eta$.

This demonstrates a deep relation between the electron density and the bosonic fields.

(b) Show that the field $\psi_\eta(x)$ generates coherent states $\psi_\eta(x)|\vec{N}\rangle_0$, i.e., eigenstates of $b_{q\eta}$. Thus, it can be represented as

$$\psi_\eta(x)\,|\vec{N}\rangle_0 = e^{\sum_{q>0} \alpha_q(x) b_{q\eta}^\dagger}\, F_\eta \hat{\lambda}_\eta(x)\,|\vec{N}\rangle_0, \tag{14}$$

where $\hat{\lambda}_\eta(x)$ is a phase operator. Determine $\alpha_q(x)$ and $\hat{\lambda}_\eta(x)$.

(c) This scheme can be generalized to an arbitrary state $|\vec{N}\rangle$ with multiple particle-hole excitations,

$$|\vec{N}\rangle = f(\{b_{q\eta}^\dagger\})|\vec{N}\rangle_0, \tag{15}$$

where f is a well-behaved function. Using the following two identites

$$\psi_\eta(x) f(\{b_{q\eta'}^\dagger\}) = f(\{b_{q\eta'}^\dagger - \delta_{\eta\eta'}\alpha_q^*(x)\})\psi_\eta(x), \tag{16}$$

$$f(\{b_{q\eta'}^\dagger - \delta_{\eta\eta'}\alpha_q^*(x)\}) = e^{-i\varphi_\eta(x)} f(\{b_{q\eta'}^\dagger\}) e^{i\varphi_\eta(x)}, \tag{17}$$

and that the Klein factors commute with the bosonic operators $b_{q\eta}^{(\dagger)}$, to verify from former results on this problem set the so-called **bosonization identity**

$$\psi_\eta(x)\,|\vec{N}\rangle = F_\eta \hat{\lambda}_\eta(x) e^{-i\varphi_\eta^\dagger(x)} e^{-i\varphi_\eta(x)}\,|\vec{N}\rangle. \tag{18}$$

Exercise 4. *Fermion and boson Green's functions*

We consider the Tomonaga–Luttinger Hamiltonian (1) and determine the imaginary-time Green's function in the bosonic and fermionic representation.

(a) First, calculate the imaginary-time fermionic Green's function in the limit $L \to \infty$ for $T \neq 0$ (defined only for $\tau \neq 0$):

$$-G_{\eta\eta'}(\tau, x) = \langle \mathcal{T} \psi_\eta(\tau, x) \psi^\dagger_{\eta'}(0,0) \rangle = \Theta(\tau) G^>_{\eta\eta'}(\tau, x) + \Theta(-\tau) G^<_{\eta\eta'}(\tau, x), \tag{19}$$

where

$$\begin{aligned} -G^>_{\eta\eta'}(\tau, x) &= \langle \psi_\eta(\tau, x) \psi^\dagger_{\eta'}(0,0) \rangle, \\ G^<_{\eta\eta'}(\tau, x) &= \langle \psi^\dagger_{\eta'}(0,0) \psi_\eta(\tau, x) \rangle. \end{aligned}$$

Hint: After Fourier transformation and taking the limit $L \to \infty$, perform a contour integral and close the semicircle in the lower (upper) half plane of the complex plane for $x > 0$ ($x < 0$). It is necessary to introduce a regularizer a by including explicitly the factor $e^{-k\,\mathrm{sgn}\,(\tau) a}$ to ensure convergence of the integral in the limit $\tau \to 0$.

(b) Verify that the fermionic correlator shows an algebraic decay in the zero-temperature limit:

$$G_{\eta\eta'}(\tau, x) \xrightarrow[T \to 0]{} -\frac{\delta_{\eta\eta'}}{\tau + ix + \mathrm{sgn}\,(\tau) a}. \tag{20}$$

(c) It can be shown that the fermionic and bosonic Green's functions, for a Hamiltonian quadratic in the bosonic representation [cf. Eq. (9)], are related by

$$G_{\eta\eta'}(\tau, x) \xrightarrow[T \to 0]{} -\delta_{\eta\eta'} a^{-1}\, \mathrm{sgn}\,(\tau) e^{\langle \mathcal{T} \phi_\eta(\tau, x) \phi_\eta(0,0) - \phi_\eta(0,0)\phi_\eta(0,0) \rangle}, \tag{21}$$

where

$$\phi_\eta(x) = \varphi^\dagger_\eta(x) + \varphi_\eta(x) = -\sum_{q>0} \frac{1}{\sqrt{m_q}} \left(e^{-iqx} b_{q\eta} + e^{iqx} b^\dagger_{q\eta} \right) e^{-qa/2}.$$

Determine the bosonic imaginary-time Green's function

$$-\mathcal{G}_{\eta\eta'}(\tau, x) = \langle \mathcal{T} \phi_\eta(\tau, x) \phi_{\eta'}(0,0) \rangle = \Theta(\tau) \mathcal{G}^>_{\eta\eta'}(\tau, x) + \Theta(-\tau) \mathcal{G}^<_{\eta\eta'}(\tau, x), \tag{22}$$

where

$$\begin{aligned} -\mathcal{G}^>_{\eta\eta'}(\tau, x) &= \langle \phi_\eta(\tau, x) \phi_{\eta'}(0,0) \rangle, \\ -\mathcal{G}^<_{\eta\eta'}(\tau, x) &= \langle \phi_{\eta'}(0,0) \phi_\eta(\tau, x) \rangle, \end{aligned}$$

for $L \neq \infty$ and $T = 0$.

(d) Finally, show that the result for the bosonic Green's function (22), when substituted into Eq. (21), is in agreement with the result for the fermionic Green's function (20).

Discussion of the problem set on Dec. 10 and Dec. 17, 2019.