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Problem Set 4: Particle in a dissipative environment and Grassmann calculus

Exercise 1. Particle in a dissipative environment

We consider the influence of a non-interacting bosonic bath on a single particle in a potential V. The full action reads

$$S = S_{\text{bath}} + S_{\text{particle}} + S_{\text{coupling}}, \tag{1}$$

where the terms

$$S_{\text{bath}} = \int \mathrm{d}t \sum_{\alpha} \frac{m_\alpha}{2}\big(\dot{q}_\alpha^2 - \omega_\alpha^2 q_\alpha^2\big), \quad S_{\text{particle}} = \int \mathrm{d}t \left(\frac{m}{2}\dot{q}^2 - V(q)\right), \tag{2}$$

describe the bath and the particle, respectively. Here, q is the degree of freedom of the particle, and q_α are the degrees of freedom of the bath. We assume that the bath has a temperature T, and the coupling is linear in q_α:

$$S_{\text{coupling}} = -\int \mathrm{d}t \sum_{\alpha} \left(c_\alpha q_\alpha q + \frac{(c_\alpha q)^2}{2 m_\alpha \omega_\alpha^2}\right). \tag{3}$$

Here, c_α describes the strength of the interaction between a mode α and the particle. Note that: (i) The second term of order $O(q^2)$ was added to S_{coupling} for convenience, in order to eliminate an effective change of $V(q)$ caused by the bath. (ii) We consider the quantum mechanics of a single particle; therefore, second quantization is not needed (but do not forget that the bath is bosonic).

(a) Give an expression for the partition function Z by using the Eucleadean action [i.e., by using imaginary time $\tau \in (0, \beta)$].

(b) Integrate out the bath, i.e., perform the Gaussian integrals over the degrees of freedom of the bath q_α. You should find the effective action

$$S_{\text{eff}} = S_{\text{particle}} + \frac{1}{2T} \sum_{\omega_n, \alpha} \frac{\omega_n^2 c_\alpha^2 q(\omega_n) q(-\omega_n)}{m_\alpha \omega_\alpha^2 (\omega_\alpha^2 + \omega_n^2)}. \tag{4}$$

Is ω_n in this expression fermionic or bosonic?

(c) Rewrite the influence of the bath as $-T \int_0^\beta \mathrm{d}\tau \mathrm{d}\tau' q(\tau) K(\tau - \tau') q(\tau')$ and determine K from the above expression for S_{eff}.

(d) Show that, in the case of Ohmic dissipation with $K(\omega_n) = \frac{\eta}{2}|\omega_n|$ and for $\tau \ll T^{-1}$, we obtain

$$K(\tau) \simeq \frac{\eta}{2\pi T \tau^2}. \tag{5}$$

(e) Let us now specify the shape of the potential and consider

$$V(q) = \begin{cases} m\omega_c^2 q^2/2, & 0 < |q| < a, \\ -\infty, & |q > a, \end{cases}$$

with Ohmic dissipation in the above limit. Compute the action of the bounce solution, i.e., a non-trivial solution of the equations of motion of the inverted potential. How do the parameters η and T influence the action of the bounce? What does this mean physically?

Exercise 2. Grassmann calculus

Let $\eta_i, \overline{\eta}_i, \xi_i, \overline{\xi}_i$ be Grassmann numbers, and A a complex matrix. Summation conventions are used in this exercise.

1. Show that under a change of coordinates $\eta_i = A_{ij}\xi_j$ one obtains

$$\int d\eta_N \ldots d\eta_1 f(\eta_1, \ldots, \eta_N) = \int d\xi_N \ldots d\xi_1 f(A_{1j}\xi_j, \ldots, A_{Nj}\xi_j)(\det A)^{-1}. \tag{6}$$

2. Show the following identities:

$$\int \left(\prod_{i=1}^{N} d\overline{\eta}_i d\eta_i \right) \exp\left(-\overline{\eta}_i A_{ij} \eta_j\right) = \det A, \tag{7}$$

$$\int \left(\prod_{i=1}^{N} d\overline{\eta}_i d\eta_i \right) \exp\left(-\overline{\eta}_i A_{ij} \eta_j + \overline{\xi}_i \eta_i + \overline{\eta}_i \xi_i\right) = \det A \cdot \exp\left(\overline{\xi}_i \left(A^{-1}\right)_{ij} \xi_j\right), \tag{8}$$

$$\int \left(\prod_{i=1}^{N} d\overline{\eta}_i d\eta_i \right) \eta_n \overline{\eta}_m \exp\left(-\overline{\eta}_i A_{ij} \eta_j\right) = \det A \left(A^{-1}\right)_{nm}. \tag{9}$$

To show (8) and (9), assume that A is Hermitian. Generalization of the last result leads to Wick's theorem for fermions.

Discussion of the problem set on Nov. 12, 2019.