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Problem Set 2: Path Integrals

Exercise 1. Lagrangian formulation of the path integral

In the lecture, we derived the path integral formula for the transition amplitude in the Hamiltonian formalism:

$$G(x_b, t_b; x_a, t_a) = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}\{x, p\} \exp\left(\frac{i}{\hbar} S[x(t), p(t)]\right), \tag{1}$$

$$\text{where} \quad \mathcal{D}\{x,p\} = \lim_{N\to\infty} \prod_{i=1}^{N-1} dx_i \prod_{j=1}^{N} \frac{dp_j}{2\pi\hbar}, \quad S = \int_{t_a}^{t_b} dt\,[p\dot{x} - H(p,x)]\,.$$

Let's consider a Hamiltonian $H = p^2/2m + V(x)$. By applying a time-slicing and integrating out momenta, show that G can be rewritten as

$$G(x_b, t_b; x_a, t_a) = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \tilde{\mathcal{D}}x \exp\left(\frac{i}{\hbar} S[x(t), \dot{x}(t)]\right), \tag{2}$$

$$\text{where} \quad \tilde{\mathcal{D}}x = \lim_{N\to\infty} \left(\frac{mN}{2\pi\hbar i(t_b - t_a)}\right)^{N/2} \cdot \prod_{i=1}^{N-1} dx_i, \qquad S = \int_{t_a}^{t_b} dt \left[\frac{m\dot{x}^2}{2} - V(x)\right].$$

You may apply a formula for Gaussian integrals, which holds for imaginary matrices by analytic continuation (*Fresnel integral formula*).

Exercise 2. Path integral for the harmonic oscillator

Consider the transition amplitude for the harmonic oscillator, given by (with $\hbar = 1$):

$$G(x_b, t_b; x_a, 0) = \int_{x(0)=x_a}^{x(t_b)=x_b} \tilde{\mathcal{D}}x \exp(iS)\,, \quad \text{with } S = \int_0^{t_b} dt \frac{m}{2}(\dot{x}^2 - \omega^2 x^2). \tag{3}$$

(a) Let x_{cl} be a classical trajectory with $x_{cl}(0) = x_a, x_{cl}(t_b) = x_b$. Show that

$$G(x_b, t_b; x_a, 0) = \exp(iS[x_{cl}]) \cdot G(0, t_b; 0, 0) \tag{4}$$

by choosing the parametrization $x(t) = x_{cl}(t) + y(t)$ for an arbitrary path $x(t)$.

(b) To evaluate $G(0, t_b, 0, 0)$, we perform a time-slicing. Check that the time-sliced action can be written as

$$S_N = \frac{m\epsilon}{2} \sum_{i=1}^{N} \left(\frac{y_i - y_{i-1}}{\epsilon}\right)^2 - \omega^2 y_i^2, \quad y_0 = y_N = 0, \quad \epsilon N = t_b. \tag{5}$$

(c) Let's introduce the *lattice derivatives*

$$\nabla y(t) = \frac{1}{\epsilon}\big(y(t+\epsilon) - y(t)\big), \quad \overline{\nabla} y(t) = \frac{1}{\epsilon}\big(y(t) - y(t-\epsilon)\big). \tag{6}$$

Rewrite the kinetic part of S_N in terms of the *lattice laplacian* $\overline{\nabla}\nabla$. What is the matrix structure $(\overline{\nabla}\nabla)_{ij}$ of this operator?

(d) Compute $(-\overline{\nabla}\nabla y)(\nu)$ in Fourier space (for a continuous function y).

(e) For the discretized path y_i, the Fourier transform can be phrased as

$$y_i = \sum_{n=1}^{N-1} \sqrt{\frac{2}{N}} \sin(\nu_n t_i) y(\nu_n), \quad t_i = i\epsilon \ (i \in \mathbb{N} \text{ here}), \quad \nu_n = \frac{\pi n}{\epsilon N}. \tag{7}$$

Conclude the eigenvalues of the matrix $(-\overline{\nabla}\nabla - \omega^2)_{ij}$, and evaluate $G(0, t_b; 0, 0)$.

(f) Use the substitution

$$\sin\frac{\epsilon\tilde{\omega}}{2} := \frac{\epsilon\omega}{2} \tag{8}$$

and the identities

$$\prod_{n=1}^{N-1}\left[1 - \frac{\sin^2 x}{\sin^2 \frac{n\pi}{2N}}\right] = \frac{1}{\sin 2x}\frac{\sin(2Nx)}{N} \tag{9}$$

and

$$\prod_{n=1}^{N-1}\left[1 + x^2 - 2x\cos(\tfrac{n\pi}{N})\right] = \frac{x^{2N}-1}{x^2-1} \tag{10}$$

to show that (assuming $t_b \cdot \tilde{\omega} < \pi$)

$$G(0, t_b; 0, 0) = \lim_{\epsilon\to 0} \frac{1}{\sqrt{2\pi i/m}} \sqrt{\frac{\sin\tilde{\omega}\epsilon}{\epsilon \sin\tilde{\omega} t_b}}. \tag{11}$$

(g) Take the limit $\epsilon \to 0$.

(h) Determine $G(0, t_b; 0, 0)$ for a free particle by taking the limit $\omega \to 0$.

(i) Compute the euclidean transition amplitude $G(0, -i\tau; 0, 0)$ in the limit $\tau \to \infty$.

Exercise 3. Classical interpretation of instantons

(a) In the lecture we have found that an instanton and anti-instanton attract each other. Find a qualitative explanation of this which is based on the picture of a particle moving in the inverted double well potential.

(b) Let's consider a periodic potential where there can be multiple instanton configurations. How do these interact?

No computations are required in this exercise!

Discussion of the problem set on Oct. 29, 2019.