Exercises for Conformal Field Theory (MD4)

1 Solution exercise 1

We have $N(TT)_n = \sum_{k>-2} L_{n-k}L_k + \sum_{k\leq -2} L_k l_{n-k}$ and $(\partial^2 T)_n = (n+2)(n+3)L_n$. We compute

$$\begin{aligned}
[L_m, \mathcal{N}(TT)_n] &= \left[L_m, \sum_{k>-2} L_{n-k}L_k + \sum_{k\leq -2} L_k l_{n-k} - \frac{3}{10}(n+2)(n+3)L_n\right] \\
&= \underbrace{-\frac{3}{10}(n+2)(n+3)(m-n)L_{m+n}}_{\equiv I} + \sum_{k\geq -1} L_{n-k}L_{m+k}(m-k) + (m-n+k)L_{m+n-k}L_k \\
&\qquad + \sum_{k\leq -2} L_k L_{m+n-k}(m-n+k) + (m-k)L_{m+k}L_{n-k} \\
&= I + \sum_{k\geq -1}(m-n+k)L_{m+n-k}L_k + \sum_{k\leq -2} L_k L_{m+n-k}(m-n+k) \\
&\qquad \sum_{\tilde{k}\geq m-1}(2m-\tilde{k})L_{m+n-\tilde{k}}L_{\tilde{k}} + \sum_{\tilde{k}\leq -2+m}(2m-\tilde{k})L_{\tilde{k}}L_{m+n-k}
\end{aligned} \tag{1}$$

In the third equality we introduced a new summation index $\tilde{k} = m+k$. Then we have $k = \tilde{k} - m$ which gives the last two terms in the above expression. From here we do a case by case computation.

$m = 0$

$$\begin{aligned}
&= -\frac{3}{10}(n+2)(n+3)(-n)L_n + (-n)\left(\sum_{k\geq -1} L_{n-k}L_k + \sum_{k\leq -2} L_k L_{n-k}\right) \\
&= (-n)\mathcal{N}(TT)_n
\end{aligned} \tag{2}$$

as the summations involving k in the summands cancel.

$m = 1$

$$\begin{aligned}
&= I + (1-n)N(TT)_{n+1} + \sum_{k\geq -1} kL_{1+n-k}L_k + \sum_{k\leq -2} kL_k L_{1+n-k} \\
&\qquad + \sum_{\tilde{k}\geq 0}(2-\tilde{k})L_{1+n-\tilde{k}}L_{\tilde{k}} + \sum_{\tilde{k}\leq -1}(2-\tilde{k})L_{\tilde{k}}L_{1+n-k} \\
&= I + (1-n)N(TT)_{n+1} - L_{2+n}L_{-1} + L_{-1}L_{2+n} - 2L_{2+n}L_{-1} + 2L_{-1}L_{2+n} + 2N(TT)_{1+n} \\
&= -\frac{3}{10}(n+3)(n+4)(3-n)L_{1+n} + (3-n)N(TT)_{1+n} + \frac{3}{10}\left[(n+3)(n+4)(3-n)\right. \\
&\qquad \left. -(1-n)(n+2)(n+3)\right]L_{1+n} - 3\left[L_{2+n}, L_{-1}\right] \\
&= (3-n)\mathcal{N}(TT)_{1+n} + \frac{3}{10}\left[10n+30\right]L_{1+n} - (3n+3)L_{1+n} \\
&= (3-n)\mathcal{N}(TT)_{1+n}
\end{aligned} \tag{3}$$

The case $m = -1$ has the same steps as the $m = 1$ case.