Exercises for Conformal Field Theory (MD4)

Problem set 10, due January 15, 2020

If you have questions write an E-mail to: mtraube@mpp.mpg.de

1 $c = 1$ theories and simple currents

In the lecture you encountered Schellekens and Yankielowicz simple current method to generate modular invariant partition functions. In an example you constructed the D-invariants from the ADE classification of $\widehat{\mathfrak{su}}(2)_k$ conformal field theories.

Also for the $c = 1$ theories an ADE classification is known. An immediate question arises: Can one similarly relate the A and D-series of the $c = 1$ theories by using simple currents? We will address this question in the following exercise. A good starting point is certainly the tensor product of two Ising models $\mathrm{Vir}_{\frac{1}{2}} \otimes \mathrm{Vir}_{\frac{1}{2}}$ having $c = 1$.

Recall: A single Ising model $\mathrm{Vir}_{\frac{1}{2}}$ is the theory of a single fermion and has three primaries $1, \sigma, \psi$ with $h_1 = 0, h_\sigma = \frac{1}{16}$ and $h_\psi = \frac{1}{2}$. The nontrivial fusion rules are

$$[\psi] \times [\psi] = [1], \qquad [\sigma] \times [\psi] = [\sigma], \qquad [\sigma] \times [\sigma] = [1] + [\psi] \tag{1}$$

while 1 acts trivially as identity.

A) Recall that one can always write down the trivial modular invariant partition function $\mathcal{Z}_1 = \chi_i M_{ij} \overline{\chi}_j$ where the characters from the left and right are connected by the identity matrix $M_{ij} = \delta_{ij}$. Write down $\mathcal{Z}_1 = \chi_i M_{ij} \overline{\chi}_j$ and recall exercise sheet 8 to verify that we actually sit at the D-branch of the ADE classification of $c = 1$ theories.

B) In $\mathrm{Vir}_{\frac{1}{2}}$ there is one nontrivial simplecurrent $\psi \in \mathrm{Vir}_{\frac{1}{2}}$. Therefore also $\psi_1 \psi_2 \in \mathrm{Vir}_{\frac{1}{2}} \otimes \mathrm{Vir}_{\frac{1}{2}}$ is a simple current as you can check by writing down the fusion rules. Construct the modular invariant partition function for this simple current. Verify that we ended up with the partition function of the $\widehat{\mathfrak{so}}(2)_1$ theory in the A-series of the ADE classification of $c = 1$ theories.

C) We have established the way from the D to the A series but one usually takes the A series as diagonal invariant. To do so take the result from last exercise and interpret it as diagonal invariant in terms of characters of new highest weights. Give the conformal dimension of every character by expanding the character. Explain from the expansion why we actually need four instead of three characters.

D) Write the characters in terms of θ functions and find a intuitive way to distinguish the two equivalent characters. State the modular S-matrix and deduce the fusion rules, possibly with help of a small computer program to speed up the matrix multiplications. *Hint: Some of the results already appeared in the lecture*

That the simple currents are able to mimic the orbifold procedure in this example is of course not a coincident.