Exercises for Conformal Field Theory (MD4)

Problem set 7, due December 11, 2019

If you have questions write an E-mail to: mtraube@mpp.mpg.de

1 The Poisson Resummation Formula

To show the modular invariance of e.g. the partition function of free boson on a circle one needs the Poisson resummation formula

$$\sum_{n\in\mathbb{Z}} \exp\left(-\pi a n^2 + bn\right) = \frac{1}{\sqrt{a}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{\pi}{a}\Big(k + \frac{b}{2\pi i}\Big)^2\right) . \tag{1}$$

Its key property is to invert the prefactor which is needed to compute the behavior under a modular S-transformation. You are asked to verify this formula in two different ways:

A) Use the discrete Fourier transform of the Dirac comb $\sum_n \delta(x-n)$

$$\sum_{n\in\mathbb{Z}} \delta(x-n) = \sum_{k\in\mathbb{Z}} e^{2\pi i k x} . \tag{2}$$

B) The inversion property of the Poisson resummation formula is of course well known from the Fourier transformation. Indeed one can easily check that both sides in (1) are connected by a Fourier transformation. This hints that the above formula is a corollary of the more general formula

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{k\in\mathbb{Z}} \hat{f}(k), \text{ where } \hat{f}(z) = \int_{-\infty}^{\infty} f(x) e^{2\pi i z x}\, dx.$$

Prove that this formula holds for any test function f. Do not use the Dirac comb method again!

Hints: At some place you might need a shift by $i\epsilon$. Furthermore any test function f has a Riemann-Hilbert decomposition $f(z) = f_+(z) + f_-(z)$ where $f_\pm(z)$ is analytically (singular freely) extendable to the upper/lower plane.

2 The Free Boson on a Circle of Radius $R = \sqrt{2k}$

You have learned in the lecture that the partition function of a free boson on a circle of radius $R = \sqrt{2k}$ where $k \in \mathbb{Z}^+$ can be written as

$$\mathcal{Z}(\tau,\overline{\tau}) = \frac{1}{|\eta(\tau)|^2} \sum_{m=-k+1}^{k} \left|\Theta_{m,k}(q)\right|^2 . \tag{3}$$

The Θ functions are defined as

$$\Theta_{m,k}(\tau) := \sum_{n\in\mathbb{Z}+\frac{m}{2k}} q^{kn^2} , \qquad\qquad -k+1 \le m \le k . \tag{4}$$

(This is the generalized Θ function from the last sheet when ignoring the angular momentum dependence.)

A) Compute the transformation property of the Θ function under a modular S-transformation. Write it in the form

$$\Theta_{m,k} = \sqrt{-i\tau} \sum_{m'=-k+1}^{k} S_{m,m'} \Theta_{m',k}(\tau) . \tag{5}$$

B) Explain why the matrix S in (5) is really the modular S matrix. To do so state the characters $\chi_m^{(k)}$ of the theory and write their transformation law using (5).

C) Take a character $\chi_m^{(k)}$ and expand it in powers of q. Deduce the conformal dimension of the highest weight corresponding to the character.

Note: Exercise B) and C) can be done without the solution of exercise (i). Everything you computed in this exercise will reappear in $\mathcal{N} = 2$ superconformal field theories since their minimal models can be written as cosets with several $\hat{\mathfrak{u}}(1)_k$ factors.