Exercises for Conformal Field Theory (MD4)

Problem set 3, due November 13, 2019

If you have questions write an E-mail to: mtraube@mpp.mpg.de

1 Projection to quasi-primary fields

During the lecture, we have noted that normal ordered products (NOPs) of quasi-primary fields do not need to be quasi-primary. For instance, one can easily check that the normal ordered product $N(TT)(z)$ of the chiral energy-momentum tensor $T(z)$ is not quasi-primary. However, it turns out that such NOPs can be *projected* to quasi-primary fields. For the energy-momentum tensor, the corresponding quasi-primary field reads

$$\mathcal{N}(TT) = N(TT) - \frac{3}{10}\,\partial^2 T\,. \tag{1}$$

Proof this statement, namely that (1) is indeed quasi-primary by computing the commutator with L_m.

2 Conformal family and Generating Function

According to the operator-state corresponding each conformal primary $\phi(z)$ of conformal dimension h gives rise to a state

$$|\phi\rangle = \lim_{|z|\to 0} \phi(z)\,|0\rangle = \phi_{-h}|0\rangle \tag{2}$$

called a highest weight state. By taking derivatives $\partial^k\phi$ and/or normal ordered products with T one can build infinitely many more *descendant fields* of ϕ. Together they form the *conformal family*

$$[\phi(z)] := \left\{\phi, \partial\phi, \partial^2\phi, \dots, N(T\phi), \dots\right\} \quad . \tag{3}$$

A) (optional warm-up exercise) Let us first rederive the dictionary between the field and the mode description of the conformal family. Show that

$$|\partial\phi\rangle = L_{-1}\phi_{-h}|0\rangle, \quad |N(T\phi)\rangle = L_{-2}\phi_{-h}|0\rangle\,, \quad |N(\partial T\phi)\rangle = L_{-3}\phi_{-h}|0\rangle \tag{4}$$

Explain why the first one is expected. Determine the level of these states.

In the lecture you learned that the conformal family (3) is equivalent to having states

$$\{L_{-k_1}\dots L_{-k_n}\phi_{-h}\,|0\rangle \mid k_i \geq k_{i+1},\, k_n \geq 1\} \quad . \tag{5}$$

We want to show, that the above is in fact a highest weight representation of the Virasoro-algebra. The main input is well-definedness at infinite past, i.e. at $z = 0$.

B) Consider a state

$$L_{j_1}\dots L_{j_m} L_{-k_1}\dots L_{-k_n}\phi_{-h}\,|0\rangle\,, \quad k_i \geq k_{i+1}, k_n \geq 1, j_i \geq 1 \quad . \tag{6}$$

Determine a condition on j_i, k_l s.th. the above state is non-vanishing.

Hint: Start with $m = 1$ and recall $L_n\phi_{-h}\,|0\rangle = 0$ for $n \geq 1$.

C) Consider a state

$$L_{-k_1} \dots L_{-k_n} \phi_{-h} |0\rangle \,, \quad k_i \geq 1. \tag{7}$$

Use the Virasoro-algebra to show that this state can be written as a linear combination of states in (5).

D) Conclude that (5) spans a highest representation of the Virasoro-algebra.

One of the main themes will be to determine for which vlaues of h, c this highest weight representation is irreducible, which in turn is equivalent to absence of null vectors. For $c < 1$ these are so called *minimal models.* We will come back to this on the following exercise sheets. Clearly the above vector space is infinite dimensional. But at fixed level there are only finitely many states. Let $P(N)$ be the number of partitions for a positive integer N, i.e. the number of possibilties to write N as a sum of natural numbers.

E) Convience yourself that at level N there are $P(N)$ states in the conformal family.

F) (optional) Often it is convenient to express partitions $P(N)$ in terms of generating functions. Show

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{N=0}^{\infty} P(N) q^N \quad . \tag{8}$$

Thus $\prod_{n=1}^{\infty} \frac{1}{1-q^n}$ is a generating function for $P(N)$.

G) Let $V(h,c) = \text{span}_{\mathbb{C}} \{L_{-k_1} \dots L_{-k_n} \phi_{-h} |0\rangle \mid k_i \geq k_{i+1},\, k_n \geq 1\}$, compute the *character*:

$$\chi_{(h,c)} = Tr_{V_{(h,c)}} q^{L_0 - \frac{c}{24}} \quad . \tag{9}$$

You will see this result again, when we consider the torus partition function of a free boson. In general knowing the number of states at a fixed level and the corresponding generating function is very useful when computing partition functions.

H) As you will see soon, the theory of a single free fermion ψ is a CFT as well. The conformal dimension of ψ is $h = \frac{1}{2}$ such that

$$\psi(z) = \sum_r \psi_r z^{-r-\frac{1}{2}} \,. \tag{10}$$

Actually, there are two sectors corresponding to the choice the behavior under rotations. The first sector is the Ramond (R) sector $r \in \mathbb{Z}$ while the second sector is the Neveu-Schwarz (NS) sector $r \in \mathbb{Z} + \frac{1}{2}$. Here we will only work with the NS-sector therefore the modes ψ_r are half integer valued $r = \dots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots$. The generating function of the NS-sector will be derived later in the lecture. One finds

$$\prod_{0 \leq s} \left(1 + q^{s+\frac{1}{2}}\right) = \sum_{N \in \frac{1}{2}\mathbb{Z}} P(N) q^N \tag{11}$$

where again $P(N)$ counts the states at a level N. Read off the number of states at the, say seven lowest levels. Identify the states explicitly.