Exercises for Conformal Field Theory (MD4)

Problem set 2, due November 6, 2019

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1 Equivalence of OPE and Algebra

Show along the lines of the lecture the equivalence between the OPE

$$T(z)\phi(w,\overline{w}) = \frac{h}{(z-w)^2}\phi(w,\overline{w}) + \frac{1}{z-w}\partial_w\phi(w,\overline{w}) + reg. \tag{1}$$

and the algebra

$$[L_m, \phi_n] = \big((h-1)m - n\big)\phi_{m+n}\,. \tag{2}$$

$\phi(z,\overline{z})$ denotes a conformal primary, ϕ_n its Laurent modes and h its conformal dimension. $T(z)$ is the energy-momentum tensor and L_m its Laurent modes, thus the Virasoro generators. If (2) holds only for the global transformations $m = -1, 0, 1$, ϕ is only a quasi-primary. Compare (2) to the Virasoro algebra.

2 The Energy-Momentum Tensor

Recall the Schwarzian derivative formula

$$T(z) \to T'(z) = \left(\frac{\partial f}{\partial z}\right)^2 T(f(z)) + \frac{c}{12}S(f(z),z) \tag{3}$$

where $S(w,z)$ is called the Schwarzian derivative

$$S(w,z) = \frac{1}{(\partial_z w)^2}\left((\partial_z w)(\partial_z^3 w) - \tfrac{3}{2}(\partial_z^2 w)^2\right). \tag{4}$$

A) Take an infinitesimal transformation $f(z) = z + \epsilon(z)$ and compute

$$\delta_\epsilon T(z) = T'(z) - T(z) \tag{5}$$

in first order in ϵ. You should get

$$\delta_\epsilon T(z) = \frac{c}{12}\partial_z^3\epsilon(z) + 2T(z)\partial_z\epsilon(z) + \epsilon(z)\partial_z T(z). \tag{6}$$

Which part comes from the Schwarzian derivative?

As you have learned in the lecture, the current of the conformal symmetry is $j = \epsilon(z)T(z)$ and the conserved charge therefore reads

$$Q = \frac{1}{2\pi i}\int_{\mathcal{C}} dz\; T(z)\epsilon(z) \tag{7}$$

when neglecting the antiholomorphic piece. Then for any operator $\mathcal{O}(w)$

$$\delta_\epsilon\mathcal{O}(w) = [Q, \mathcal{O}(w)] = \frac{1}{2\pi i}\int_{\mathcal{C}} dz\; \epsilon(z)[T(z), \mathcal{O}(w)] = \frac{1}{2\pi i}\int_{\mathcal{C}} dz\; \epsilon(z)\; \mathcal{R}(T(z), \mathcal{O}(w)) \tag{8}$$

where $\mathcal{R}$ denotes the Radial ordering, thus the OPE.

B) Derive the OPE $T(z)T(w)$ by comparing (8) with (6).

In the lecture the equivalence of the OPE $T(z)T(w)$ and the Virasoro algebra was shown. When going through this computation you can see that the term coming from the Schwarzian derivative gives the central extension of the Virasoro algebra. As you can easily check, the central extension vanishes for $n = -1, 0, 1$, making $T(z)$ a quasi primary.

C) Recall what this means geometrically and deduce a relation about the Schwarzian derivative (you do not need to prove this relation).

D) In which case is $T(z)$ a full conformal primary? What does this imply quantum mechanically for the energy-momentum tensor?

3 Gauge-fixing the conformal group

In two-dimensional flat euclidean space $\mathbb{R}^2 \simeq \mathbb{C}$ there are infinitely many generators of *local* conformal transformations

$$l_n = -z^{n+1}\partial_z \,, \qquad n \in \mathbb{Z} \,. \tag{9}$$

A) Determine which transformations are globally well-defined on the two sphere $S^2 \cong \mathbb{C} \cup \{\infty\}$.

B) Show that the globally well-defined generators generate

$$\begin{aligned} \mathrm{SL}(2,\mathbb{C}) : S^2 &\to S^2 \\ z &\mapsto \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C},\, ad-cb=1 \end{aligned} \tag{10}$$

(Hint: You may want to consult excercise 1 on sheet 1.)

C) $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$ are the global conformal transformations of S^2. Let $\{z_0^i, z_1^i, z_2^i\}$ with $i = 1, 2$ and $z_j^i \neq z_j^k$ for $i \neq k$ be points on S^2. Show that there is an element in $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$ mapping $\{z_0^1, z_1^1, z_2^1\} \to \{z_0^2, z_1^2, z_2^2\}$.
(Hint: You may want to show that any three points can be mapped to $\{0, 1, \infty\}$).