String Theory I

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Exercise sheet 8

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Exercise 1: Unoriented strings

As mentioned in class, the string theory with the quanta of the Kalb-Ramond antisymmetric tensor field among the massless closed string excitations is a theory of oriented strings. One can define a theory of unoriented strings by introducing an operator Ω that reverses the orientation of the strings. The theory of unoriented strings is obtained by restricting the oriented string spectrum to those states which are invariant under the action of Ω (which eliminates the quanta of the Kalb-Ramond field).

Concretely, for closed strings and open strings with only NN and DD boundary conditions Ω acts on the operators X^i according to

$$\Omega X^i(\tau,\sigma)\Omega^{-1} = X^i(\tau, 2\pi/\beta - \sigma) \tag{1.1}$$

(with $\beta = 1$ for closed strings and 2 for open strings). Moreover, demand that

$$\Omega x^- \Omega^{-1} = x^- \ , \quad \Omega p^+ \Omega^{-1} = p^+ \ . \tag{1.2}$$

(a) Use the string oscillator expansions[1]

$$\text{closed}: \ X^I(\tau,\sigma) = x^I + \alpha' p^I \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{e^{-in\tau}}{n}\left(\alpha_n^I e^{in\sigma} + \tilde{\alpha}_n^I e^{-in\sigma}\right), \tag{1.3}$$

$$\text{NN}: \ X^I(\tau,\sigma) = x^I + 2\alpha' p^I \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n}\alpha_n^I e^{-in\tau}\cos(n\sigma) \ , \tag{1.4}$$

$$\text{DD}: \ X^a(\tau,\sigma) = x_0^a + \frac{x_\pi^a - x_0^a}{\pi}\sigma + \sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n}\alpha_n^a e^{-in\tau}\sin(n\sigma) \ , \tag{1.5}$$

in order to calculate how Ω acts on the zero mode and creation/annihilation operators, i.e.

$$\Omega x^I \Omega^{-1},\ \Omega p^I \Omega^{-1},\ \Omega x^a_{0/\pi}\Omega^{-1},\ \Omega \alpha_n^I \Omega^{-1},\ \Omega \tilde{\alpha}_n^I \Omega^{-1},\ \Omega \alpha_n^a \Omega^{-1} \ . \tag{1.6}$$

x_0^a and x_π^a are the positions of the endpoints when the argument of X^a is equal to 0 or π, respectively.

(b) For a closed string and for an open string with only NN and DD boundary conditions, argue that (1.1) also holds for X^- and X^+.

[1] I use I in the formulas for the closed string and the open string directions with NN boundary conditions, in order not to confuse the index with the explicit factors of i. For the closed string $I = 1, \ldots, D-2$ and for the open string I enumerates all the transverse directions with NN boundary conditions.

(c) Which excitations survive at the massless level for an open string ending on a single D-brane (with NN boundary conditions for X^- and X^+), assuming that the open string ground states are invariant under Ω?

(d) Assuming that the closed string ground states are invariant under Ω, discuss which states are kept in the unoriented theory at level $N^\perp = 2$.

(e) Now consider open strings with ND and/or DN boundary conditions along some directions. Obviously, reversing the orientation of a string along an ND direction leads to a DN direction and vice versa. Thus, Ω acts on the operators $X^{r,(\mathrm{ND})}$ according to

$$\Omega X^{r,(\mathrm{ND})}(\tau,\sigma)\Omega^{-1} = X^{r,(\mathrm{DN})}(\tau,\pi-\sigma) \tag{1.7}$$

and analogously for $X^{r,(\mathrm{DN})}$. Use the string oscillator expansions

$$\mathrm{ND}: \qquad X^{r,(\mathrm{ND})} = x^r_\pi + i\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}_{\mathrm{odd}}} \frac{2}{n}\alpha^{r,(\mathrm{ND})}_{\frac{n}{2}} e^{-i\frac{n}{2}\tau}\cos\left(\frac{n}{2}\sigma\right) \ , \tag{1.8}$$

$$\mathrm{DN}: \qquad X^{r,(\mathrm{DN})} = x^r_0 + \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}_{\mathrm{odd}}} \frac{2}{n}\alpha^{r,(\mathrm{DN})}_{\frac{n}{2}} e^{-i\frac{n}{2}\tau}\sin\left(\frac{n}{2}\sigma\right) \tag{1.9}$$

in order to calculate how Ω acts on the zero mode and creation/annihilation operators, i.e.

$$\Omega x^r_{0/\pi}\Omega^{-1} \ , \Omega\alpha^{r,(\mathrm{ND})}_{\frac{n}{2}}\Omega^{-1}, \ \Omega\alpha^{r,(\mathrm{DN})}_{\frac{n}{2}}\Omega^{-1} \ . \tag{1.10}$$

Exercise 2: Virasoro algebra

In this exercise we consider the bosonic open string theory quantized in the old covariant way (i.e. the constraints are implemented on the states of the quantum theory and not already in the classical theory, as in the light-cone quantization). In this case the normal ordered Virasoro generators are defined as

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} :\alpha_{m-n}\cdot\alpha_n: \ , \tag{2.1}$$

where the scalar product uses the full 26-dimensional Lorentz metric, i.e.

$$\alpha_{m-n}\cdot\alpha_n = \eta_{\mu\nu}\,\alpha^\mu_{m-n}\alpha^\nu_n \ . \tag{2.2}$$

We want to show that the Virasoro generators (2.1) fulfill the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \tag{2.3}$$

with *central charge*

$$c = D \ . \tag{2.4}$$

(a) First show, using $[\alpha^\mu_m, \alpha^\nu_n] = m\,\eta^{\mu\nu}\delta_{m+n,0}$, that

$$[\alpha^\mu_m, L_n] = m\,\alpha^\mu_{m+n} \ . \tag{2.5}$$

(b) Write

$$L_m = \frac{1}{2}\sum_{k\geq 0} \alpha_{m-k}\cdot\alpha_k + \frac{1}{2}\sum_{k<0}\alpha_k\cdot\alpha_{m-k} \tag{2.6}$$

to show

$$\begin{aligned} [L_m, L_n] &= \frac{1}{2}\sum_{k\geq 0}(m-k)\,\alpha_{m+n-k}\cdot\alpha_k + \frac{1}{2}\sum_{k<0}(m-k)\,\alpha_k\cdot\alpha_{m+n-k} \\ &\quad +\frac{1}{2}\sum_{k\geq 0} k\,\alpha_{m-k}\cdot\alpha_{k+n} + \frac{1}{2}\sum_{k<0} k\,\alpha_{k+n}\cdot\alpha_{m-k}\ . \end{aligned} \tag{2.7}$$

(c) Now consider first the case $m+n \neq 0$ and show that

$$[L_m, L_n] = (m-n)L_{m+n}\ , \qquad (m+n\neq 0)\ . \tag{2.8}$$

(d) For $m+n=0$ and assuming $m>0$ (the other cases $m<0$ and $m=0$ work analogously), show that

$$[L_m, L_{-m}] = \sum_{k=0}^{\infty}(m-k)\alpha_{-k}\cdot\alpha_k + \sum_{k=1}^{\infty}(m+k)\alpha_{-k}\cdot\alpha_k + D\,A(m)\ , \tag{2.9}$$

with

$$A(m) = \frac{1}{2}\sum_{k=0}^{m} k(m-k)\ . \tag{2.10}$$

(e) Prove by induction the following identities

$$\sum_{k=1}^{m} k^2 = \frac{1}{6}(2m^3+3m^2+m)\ , \tag{2.11}$$

$$\sum_{k=1}^{m} k = \frac{1}{2}m(m+1)\ . \tag{2.12}$$

(f) Use your results of (d) and (e) in order to derive (2.3) also for $n=-m<0$.

For questions:
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