String Theory I

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Exercise sheet 7

Due on December 6, 2019

Exercise 1: Analytic continuation of $\zeta(s)$

The gamma function can be defined for $\mathrm{Re}(s) > 0$ via the integral $\Gamma(s) = \int_0^\infty dt\, \mathrm{e}^{-t}\, t^{s-1}$. Replace $t \to n\, t$ in this integral, and use the resulting equation to prove that

$$\Gamma(s)\, \zeta(s) = \int_0^\infty dt\, \frac{t^{s-1}}{\mathrm{e}^t - 1} \quad , \quad \mathrm{Re}(s) > 1 \, , \tag{1.1}$$

where $\zeta(s)$ is defined for $\mathrm{Re}(s) > 1$ via the sum $\zeta(s) = \sum_{n=1}^\infty n^{-s}$. Verify the small t-expansion

$$\frac{1}{\mathrm{e}^t - 1} = \frac{1}{t} - \frac{1}{2} + \frac{t}{12} + \mathcal{O}(t^2) \, . \tag{1.2}$$

Use the above equations to show that for $\mathrm{Re}(s) > 1$

$$\begin{aligned} \Gamma(s)\, \zeta(s) \quad &= \quad \int_0^1 dt\, t^{s-1} \left(\frac{1}{\mathrm{e}^t - 1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12} \right) + \frac{1}{s-1} - \frac{1}{2s} + \frac{1}{12(s+1)} \\ &\quad + \int_1^\infty dt\, \frac{t^{s-1}}{\mathrm{e}^t - 1} \, . \end{aligned} \tag{1.3}$$

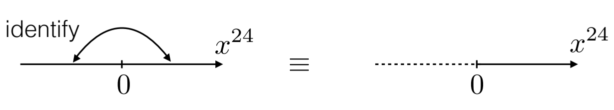
Explain why the right-hand side above is well defined for $\mathrm{Re}(s) > -2$ (except for the obvious poles at $s = -1, 0$ and 1). It follows that this right-hand side defines the analytic continuation of the left-hand side to $\mathrm{Re}(s) > -2$. Use your knowledge about the gamma function to show that $\zeta(0) = -1/2$ and $\zeta(-1) = -1/12$.

Exercise 2: Closed strings on the orbifold $\mathbb{R}^1/\mathbb{Z}_2$

The purpose of this exercise is to define closed strings in a 26-dimensional spacetime where the coordinate x^{24} is restricted to $x^{24} \geq 0$ by the identification

$$x^{24} \sim -x^{24} \, . \tag{2.1}$$

A space obtained by identifying points under transformations with fixed points is called an *orbifold*, cf. the figure for the 1-dimensional example at hand which has a fixed point at $x^{24} = 0$ (in addition in the full string theory one still has the other 25 directions spanning $\mathbb{R}^{1,24}$).



Closed string theory on an orbifold is defined by imposing a restriction on the states of closed strings living on the spacetime before orbifolding, i.e. the *parent* theory. Let us make this more precise. The string coordinate $X^{24}(\tau,\sigma)$ will be written as $X(\tau,\sigma)$, and the collection of string coordinates is therefore X^+, X^-, X^i and X, with $i = 1,\dots,23$. We introduce an operator U that implements on the string coordinates the $\mathbb{Z}_2$ transformation that defines the orbifold, i.e.

$$UX(\tau,\sigma)U^{-1} = -X(\tau,\sigma) \; , \tag{2.2}$$

as well as $UX^iU^{-1} = X^i, Up^+U^{-1} = p^+$ and $Ux^-U^{-1} = x^-$. The orbifold closed string theory keeps only U-invariant states of the parent theory.

(a) What is the U action on the operators x, p, α_n and $\tilde{\alpha}_n$ appearing in the expansion $X(\tau,\sigma) = x + \alpha' p\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n e^{-in\sigma^-} + \tilde{\alpha}_n e^{-in\sigma^+}\right)$? What is the U action on X^-? Argue that the Hamiltonian $H = L_0^\perp + \tilde{L}_0^\perp - 2$ is invariant, so that U is really a symmetry of the closed string theory.

(b) Denote the original oscillator ground states as $|k^+, \vec{k}, k\rangle$, where $\vec{k}$ is a vector with components k^i and k denotes the momentum in the 24th direction. Assume that $U|k^+, \vec{k}, 0\rangle = |k^+, \vec{k}, 0\rangle$. Show that $U|k^+, \vec{k}, k\rangle = |k^+, \vec{k}, -k\rangle$? What are the oscillator ground states of the orbifold theory?

(c) List the massless states of the orbifold theory.

It turns out that this is not the end of the story. The orbifold theory has a *twisted* sector which includes new kinds of closed strings. These closed strings can be imagined as open strings in the original space, but with the condition that the endpoints are identified by (2.1), i.e.

$$X(\tau,\sigma + 2\pi) = -X(\tau,\sigma) \; . \tag{2.3}$$

In this sector, the mode expansion for $X(\tau,\sigma)$ is different. As usual, we write

$$X(\tau,\sigma) = X_L(\sigma^+) + X_R(\sigma^-) \; . \tag{2.4}$$

(d) Use (2.3) and (2.4) to find the mode expansion for $X(\tau,\sigma)$. You may start by finding out how $X_L'(\sigma^+)$ and $X_R'(\sigma^-)$ behave when σ^+ and σ^- are increased by 2π. You should find the final result

$$X(\tau,\sigma) = i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}_{\text{odd}}}\frac{2}{n}e^{-i\frac{n}{2}\tau}\left(\alpha_{\frac{n}{2}}e^{i\frac{n}{2}\sigma} + \tilde{\alpha}_{\frac{n}{2}}e^{-i\frac{n}{2}\sigma}\right) \; . \tag{2.5}$$

The commutation relations for the modes can be obtained in a similar way as was done in exercise 2 of sheet 4. The result (which does not have to be derived) is

$$\left[\alpha_{\frac{n}{2}}, \alpha_{\frac{m}{2}}\right] = \frac{n}{2}\delta_{m+n,0} = \left[\tilde{\alpha}_{\frac{n}{2}}, \tilde{\alpha}_{\frac{m}{2}}\right] \; , \quad \left[\alpha_{\frac{n}{2}}, \tilde{\alpha}_{\frac{m}{2}}\right] = 0 \; . \tag{2.6}$$

(e) Equation (2.2) must also hold in the twisted sector. What is the U action on the new oscillators? Give the ground states in the twisted sector.

(f) Use the zeta function regularization discussed in class to find the mass-squared formula in the twisted sector. Give the masses of the ground states. Exhibit the states at the first excited level and give their masses. Where do the states in the twisted sector "live"?