String Theory I

Ludwig-Maximilians-Universität München
Dr. Michael Haack

Exercise sheet 6

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Exercise 1: Rotating open string in the light-cone gauge

Consider the motion of a string (in D dimensions with $D \geq 4$ and with NN-boundary conditions in all directions), defined by $x^- = x^i = 0$ and the vanishing of all the coefficients α_n^i with the exception of

$$\alpha_1^1 = (\alpha_{-1}^1)^* = a \quad , \quad \alpha_1^2 = (\alpha_{-1}^2)^* = ia \ . \tag{1.1}$$

Here a is a dimensionless positive constant. We want to construct a solution that represents an open string that is rotating in the (x^1, x^2)-plane (and thus has vanishing center of mass momentum).

(a) What is the mass (or energy) of this string?

(b) Construct the explicit functions $X^1(\tau, \sigma)$ and $X^2(\tau, \sigma)$. What is the length of the string in terms of a and α'?

(c) Calculate the $L_n^\perp$ modes for all n. Use your result to construct $X^-(\tau, \sigma)$.

(d) Determine the value of p^+ by demanding that $X^{D-1}(\tau, \sigma) = 0$ (as we want to describe a string rotating in the (x^1, x^2)-plane). Find the relation between t and τ. What is the angular frequency of the rotation?

(e) Express the energy and the angular frequency of the rotating string in terms of its length. How does the frequency change if one increases the energy of the string? Interpret this result.

Exercise 2: Action of $L_0^\perp - \tilde{L}_0^\perp$

In class we discussed that in the classical theory of a closed string $L_0 - \tilde{L}_0$ generates σ-translations via the Poisson-bracket. In the quantum theory the Poisson-bracket is replaced by the commutator. In light-cone gauge, this implies

$$\left[L_0^\perp - \tilde{L}_0^\perp, X^I(\tau, \sigma)\right] = i\frac{\partial X^I}{\partial \sigma} \ . \tag{2.1}$$

(a) Using (2.1) and the definition $P \equiv L_0^\perp - \tilde{L}_0^\perp$, show that

$$e^{-iP\sigma_0} X^I(\tau, \sigma) e^{iP\sigma_0} = X^I(\tau, \sigma + \sigma_0) \tag{2.2}$$

for a constant σ_0.

Hint: For two linear operators A and B, one has $e^A B e^{-A} = \sum_{m=0}^{\infty} \frac{1}{m!}[A, B]_m$ with $[A, B]_m = [A, [A, B]_{m-1}]$ and $[A, B]_0 = B$.

(b) Use the result of part (a) to calculate $e^{-iP\sigma_0}\alpha_n^I e^{iP\sigma_0}$ and $e^{-iP\sigma_0}\tilde{\alpha}_n^I e^{iP\sigma_0}$.

(c) Consider the state

$$|\phi\rangle = \alpha_{-m}^I \tilde{\alpha}_{-n}^J |p^+, \vec{p}_T\rangle \quad , \quad m, n > 0 \ , \tag{2.3}$$

where $\vec{p}_T$ stands for the transversal components of the momentum. Calculate $e^{-iP\sigma_0}|\phi\rangle$. What is the condition that makes the state $|\phi\rangle$ invariant under σ-translations?

Remark: You can assume that the ground states $|p^+, \vec{p}_T\rangle$ are invariant under σ-translations.

Exercise 3: Partition function and string entropy

In this exercise we consider a generating function for the number of string states at a fixed excitation level. For simplicity we consider an open string in light-cone quantization with DD boundary conditions along all spatial directions (i.e. the endpoints are ending on a D0-brane) so that we can ignore any spatial momentum. Its Hilbert space is spanned by the normalized states

$$|\{\lambda_{n,i}\}\rangle = \prod_{i=1}^{D-2} \prod_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_{n,i}! n^{\lambda_{n,i}}}} \left(\alpha_n^{i\dagger}\right)^{\lambda_{n,i}} |0\rangle, \tag{3.1}$$

which is characterized by the non-negative integers $\lambda_{n,i}$ denoting the number of times that the creation operator $\alpha_n^{i\dagger}$ appears. The number operator

$$N^\perp \equiv \sum_{i=1}^{D-2} \sum_{n=1}^{\infty} \alpha_n^{i\dagger} \alpha_n^i \tag{3.2}$$

acts on the states (3.1) according to

$$N^\perp |\{\lambda_{n,i}\}\rangle = \left(\sum_{i=1}^{D-2} \sum_{n=1}^{\infty} n \lambda_{n,i} \right) |\{\lambda_{n,i}\}\rangle \ . \tag{3.3}$$

To find the number of states at a given level (i.e. a given eigenvalue of the number operator (3.2)), we introduce the *partition function*

$$Z(q) = \mathrm{Tr}_{\mathscr{H}} \left(q^{N^\perp} \right) \ , \tag{3.4}$$

where the trace is over all the states of the Hilbert space, given by (3.1), and $q \in \mathbb{C}$ is an auxiliary variable with $|q| < 1$.

(a) Show that

$$Z(q) = \left(\prod_{m=1}^{\infty} (1 - q^m) \right)^{-(D-2)} \ . \tag{3.5}$$

Remark: For $D = 26$ (the critical dimension of the bosonic string), $Z(q)$ can be written in terms of the famous *Dedekind η-function*,

$$Z(q) = q\, \eta(\tau)^{-24} \ , \quad \eta(\tau) = e^{\frac{i\pi\tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}) \ , \quad q = e^{2\pi i \tau}, \quad \mathrm{Im}\tau > 0 \ . \tag{3.6}$$

(b) Argue that $Z(q)$ is a *generating function* for the number d_N of states at level N in the sense that

$$Z(q) = \sum_{N=0}^{\infty} d_N q^N \ . \tag{3.7}$$

Remark: Using properties of the Dedekind η-function, at large N the numbers d_N can be approximated by the *Hardy-Ramanujan formula*

$$d_N \approx N^{-27/4} e^{4\pi\sqrt{N}} \ . \tag{3.8}$$

(c) It follows that the statistical entropy of the string for large energy E (which is equal to the rest mass M for a string without spatial momentum and, thus, it is proportional to $\sqrt{N}$) is given by

$$S(E) = k_B \ \log d_N \approx k_B \, 4\pi\sqrt{N} = k_B 4\pi E \sqrt{\alpha'} \ . \tag{3.9}$$

What do you infer for the temperature of the string at high energies?

Remark: A similar result holds when taking into account the momentum of the string, and also for the closed string and for the superstring. For more details, you could consult chapter 22 of the book by Zwiebach (second edition).

For questions:

michael.haack@lmu.de