String Theory I

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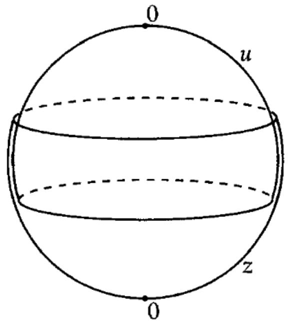
Exercise sheet 5

Due on November 22, 2019

Exercise 1: Metric on the sphere

The sphere S_2 can be covered by two coordinate patches. One can introduce complex coordinates in each patch, $z = z_1 + iz_2$ and $u = u_1 + iu_2$ and take as the open neighborhoods of the north and south pole the open disks $|z| < \rho$ and $|u| < \rho$, with $\rho > 1$. The transition function in the overlap region can be taken to be

$$u = 1/z \; . \tag{1.1}$$



A globally defined metric for a sphere of radius r is given by

$$ds^2 = \frac{4r^2 dz d\bar{z}}{(1+|z|^2)^2} = \frac{4r^2 du d\bar{u}}{(1+|u|^2)^2} \; , \tag{1.2}$$

(cf. exercise 2, sheet 2, for $r = 1$). This metric has constant curvature $R = 2/r^2$ (leading to the Euler number $\chi = \frac{1}{4\pi}\int_{S_2}\sqrt{g}R = 2$).

(a) Show that (1.2) is indeed a well defined global metric, i.e. verify that the two expressions in the u and the z coordinates agree in the overlap region.

(b) Obviously, one can perform Weyl-transformations in the z- and the u-patches separately that bring (1.2) to the flat gauge in each of the patches. Show that this does not correspond to a globally well defined Weyl-transformation though, i.e. the two Weyl-transformations do not agree in the overlap region.

Remark: Put differently, choosing the flat gauge in both patches simultaneously does not lead to a well defined global metric, as the two expressions do not agree in the overlap region.

Exercise 2: Angular momentum tensor

Show that for a closed string the angular momentum tensor

$$J^{\mu\nu} = T \int_0^{2\pi} d\sigma \, (X^\mu \, \partial_\tau X^\nu - X^\nu \, \partial_\tau X^\mu) \tag{2.1}$$

is given by

$$J^{\mu\nu} = x^\mu \, p^\nu - x^\nu \, p^\mu - i \sum_{n \neq 0} \frac{1}{n} \left(\alpha^\mu_{-n} \, \alpha^\nu_n + \tilde{\alpha}^\mu_{-n} \, \tilde{\alpha}^\nu_n \right) \ . \tag{2.2}$$

Exercise 3: DD and ND boundary conditions for open strings

(a) Consider an open string coordinate X^a with Dirichlet boundary conditions on both sides, i.e.

$$X^a(\tau, 0) = x_1^a \quad , \quad X^a(\tau, \pi) = x_2^a \ . \tag{3.1}$$

Show that the most general solution to the open string equations of motion with these boundary conditions can be expanded as

$$X^a = x_1^a + \frac{x_2^a - x_1^a}{\pi} \sigma + \sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^a e^{-in\tau} \sin(n\sigma) \tag{3.2}$$

with $\alpha^a_{-n} = (\alpha^a_n)^*$.

(b) Now consider an open string coordinate X^r with Neumann boundary conditions on one side and Dirichlet boundary conditions on the other side, i.e.

$$\frac{\partial X^r}{\partial \sigma}(\tau, 0) = 0 \quad , \quad X^r(\tau, \pi) = x_2^r \ . \tag{3.3}$$

Show that the most general solution to the open string equations of motion with these boundary conditions can be expanded as

$$X^r = x_2^r + i\sqrt{2\alpha'} \sum_{n \in \mathbb{Z}_{\text{odd}}} \frac{2}{n} \alpha^r_{\frac{n}{2}} e^{-i\frac{n}{2}\tau} \cos\left(\frac{n\sigma}{2}\right) \tag{3.4}$$

with $\alpha^r_{-\frac{n}{2}} = (\alpha^r_{\frac{n}{2}})^*$.

For questions:

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