String Theory I

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Exercise sheet 4

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Exercise 1: Closed string solution

Consider a closed string with initial conditions

$$\begin{aligned}
X^0(\tau=0,\sigma) &= 0 \quad , \quad \dot{X}^0(\tau=0,\sigma) = R \ , \\
X^1(\tau=0,\sigma) &= R\cos(\sigma) \quad , \quad \dot{X}^1(\tau=0,\sigma) = 0 \ , \\
X^2(\tau=0,\sigma) &= R\sin(\sigma) \quad , \quad \dot{X}^2(\tau=0,\sigma) = 0 \ , \\
X^i(\tau=0,\sigma) &= 0 \quad , \quad \dot{X}^i(\tau=0,\sigma) = 0 \quad , \quad i = 3,\ldots,D-1 \ , \qquad (1.1)
\end{aligned}$$

i.e. at $\tau = 0$ it is a circular string of radius R at rest, localized in the (X^1, X^2)-plane. Determine its time evolution and show that the solution solves the constraints $\dot{X} \cdot X'$ and $\dot{X}^2 + X'^2 = 0$.

Exercise 2: Virasoro generators

(a) The canonical momentum density of the Polyakov action in flat (or conformal) gauge is given by

$$P^\mu(\tau,\sigma) = \frac{\partial \mathcal{L}}{\partial \dot{X}_\mu} = T\dot{X}^\mu \ . \qquad (2.1)$$

Use the expansion of the coordinates X^μ in Fourier modes, i.e.

$$X^\mu(\tau,\sigma) = x^\mu + \alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\left[\alpha_n^\mu e^{-in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-in(\tau+\sigma)}\right] \qquad (2.2)$$

and the Poisson brackets

$$\begin{aligned}
\{X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\}_{\rm PB} &= \{P^\mu(\tau,\sigma), P^\nu(\tau,\sigma')\}_{\rm PB} = 0 \ , \qquad (2.3) \\
\{X^\mu(\tau,\sigma), P^\nu(\tau,\sigma')\}_{\rm PB} &= \eta^{\mu\nu}\delta(\sigma-\sigma') \qquad (2.4)
\end{aligned}$$

to show that

$$\{\alpha_m^\mu, \alpha_n^\nu\}_{\rm PB} = \{\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\}_{\rm PB} = -im\eta^{\mu\nu}\delta_{m+n,0} \ , \quad \{\tilde{\alpha}_m^\mu, \alpha_n^\nu\}_{\rm PB} = 0 \ , \quad m,n \in \mathbb{Z} \, . \qquad (2.5)$$

Hint: First show that

$$\begin{aligned}
\Big\{(\dot{X}^\mu \pm X^{\mu\prime})(\tau,\sigma), (\dot{X}^\nu \pm X^{\nu\prime})(\tau,\sigma')\Big\}_{\rm PB} &= \pm\frac{2}{T}\,\eta^{\mu\nu}\frac{d}{d\sigma}\delta(\sigma-\sigma') \ , \\
\Big\{(\dot{X}^\mu \pm X^{\mu\prime})(\tau,\sigma), (\dot{X}^\nu \mp X^{\nu\prime})(\tau,\sigma')\Big\}_{\rm PB} &= 0 \ . \qquad (2.6)
\end{aligned}$$

Use this to verify (2.5) (remember $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$ and $2\pi\delta(x) = \sum_{n\in\mathbb{Z}} e^{inx}$).

(b) The Virasoro generators of the closed string are given by

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n \quad , \quad \tilde{L}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n \ . \tag{2.7}$$

Use the result of part (a) to show that they satisfy the *Witt algebra*

$$\begin{aligned} \{L_m, L_n\}_{\rm PB} &= -i(m-n)L_{m+n} \ , \\ \{\tilde{L}_m, \tilde{L}_n\}_{\rm PB} &= -i(m-n)\tilde{L}_{m+n} \ , \\ \{L_m, \tilde{L}_n\}_{\rm PB} &= 0 \ . \end{aligned} \tag{2.8}$$

(c) Now use (2.4) in order to derive

$$\{\alpha_n^\mu, x^\nu\}_{\rm PB} = \{\tilde{\alpha}_n^\mu, x^\nu\}_{\rm PB} = 0 \quad (n \neq 0) \quad , \quad \{x^\mu, p^\nu\}_{\rm PB} = \eta^{\mu\nu} \ . \tag{2.9}$$

(d) In class it is discussed that the Virasoro generators generate conformal transformations via the Poisson bracket. Here you should show this explicitly, i.e.

$$\{L_n, X^\mu(\sigma^+, \sigma^-)\}_{\rm PB} = -e^{in\sigma^-} \frac{\partial}{\partial \sigma^-} X^\mu(\sigma^+, \sigma^-) \ . \tag{2.10}$$

To do so, you will need both (2.5) and (2.9).

For questions:
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