String Theory I

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Exercise sheet 3

Due on November 8, 2019

Exercise 1: Polyakov action

The Polyakov action is given by

$$S_p = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-g}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}\ . \tag{1.1}$$

As discussed in class, using

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\alpha\beta}\delta g_{\alpha\beta} \quad \text{and} \quad g^{\alpha\beta}\delta g_{\alpha\beta} = -g_{\alpha\beta}\delta g^{\alpha\beta} \tag{1.2}$$

and the general relativistic definition of the energy momentum tensor,

$$\delta S_{\text{matter}} = \frac{1}{4\pi}\int d^2\sigma\sqrt{-g}T_{\alpha\beta}\delta g^{\alpha\beta} \tag{1.3}$$

(with S_{matter} the matter part of the action, which for the string is S_p) leads to the following form of the energy momentum tensor for the Polyakov action:

$$T_{\alpha\beta} = -\frac{1}{\alpha'}\left(\partial_\alpha X\cdot\partial_\beta X - \frac{1}{2}g_{\alpha\beta}g^{\gamma\delta}\partial_\gamma X\cdot\partial_\delta X\right)\ . \tag{1.4}$$

(a) The equation of motion for the world sheet metric $g_{\alpha\beta}$ is given by $T_{\alpha\beta}=0$. Use this to show the on-shell equivalence of the Polyakov and the Nambu-Goto action.

(b) Show that the equation of motion for X^μ can be expressed as

$$g^{\alpha\beta}\nabla_\alpha(\partial_\beta X^\mu) = 0\ . \tag{1.5}$$

Hint: You will probably have to use the relation

$$\nabla_\alpha(\sqrt{-g}V^\alpha) = \partial_\alpha(\sqrt{-g}V^\alpha) \tag{1.6}$$

for an arbitrary vector V^α. Proof this (hint: eq. (1.2)).

(c) Use (1.5) to explicitly show the conservation of the energy momentum tensor (1.4), i.e.

$$\nabla^\alpha T_{\alpha\beta} = 0\ . \tag{1.7}$$

(d) Show that for any Weyl-invariant matter action the energy momentum tensor, defined by (1.3), is traceless.

(e) Now add to the Polyakov action a cosmological constant term

$$S_{cc} = \lambda\int d^2\sigma\sqrt{-g}\ . \tag{1.8}$$

Show that the equation of motion for $g_{\alpha\beta}$ implies $\lambda = 0$.

Exercise 2: Gravity in two dimensions

(a) In your GR course you learned that the symmetries of the Riemann tensor reduce its number of independent components in D dimensions to $D^2(D^2-1)/12$. This holds for any tensor $T_{\alpha\beta\gamma\delta}$ with the same symmetries as the Riemann tensor, i.e.

$$T_{\alpha\beta\gamma\delta} = -T_{\beta\alpha\gamma\delta} = -T_{\alpha\beta\delta\gamma} \ , \tag{2.1}$$

$$T_{\alpha\beta\gamma\delta} = T_{\gamma\delta\alpha\beta} \ , \tag{2.2}$$

$$T_{\alpha\beta\gamma\delta} + T_{\alpha\gamma\delta\beta} + T_{\alpha\delta\beta\gamma} = 0 \ . \tag{2.3}$$

Thus, for $D = 2$ this implies that the Riemann tensor only has one independent component, which could for example be taken to be R_{0101}. Use this result to show that in two dimensions the Riemann tensor can be completely expressed in terms of the Ricci scalar R and the metric $g_{\alpha\beta}$ as

$$R_{\alpha\beta\gamma\delta} = \frac{R}{2}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) \ . \tag{2.4}$$

(b) Compute the Einstein tensor $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ in two dimensions and interpret your result.

(c) Show that in two dimensions the combination $\sqrt{-g}R$ transforms under a Weyl rescaling $g_{\alpha\beta} \to g'_{\alpha\beta} = e^{2\omega(x)}g_{\alpha\beta}$ according to

$$\sqrt{-g}R \to \sqrt{-g'}R' = \sqrt{-g}\left(R - 2\nabla^2\omega\right) \ . \tag{2.5}$$

Note: This part of the exercise is a bit tedious.

(d) Use (2.5) to argue that the two-dimensional Einstein-Hilbert action is invariant under a Weyl transformation for a closed string worldsheet.

Remark: In contrast, for an open string world-sheet Σ with boundary $\partial\Sigma$ only the combination

$$\chi = \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{-g}R + \frac{1}{2\pi}\int_{\partial\Sigma} ds\, k \tag{2.6}$$

is invariant under Weyl transformations (ds is the proper time along the boundary). The geodesic curvature of the boundary k is defined as

$$k = \pm n_\beta t^\alpha \nabla_\alpha t^\beta \tag{2.7}$$

with t^α a unit vector tangent to the boundary and n^α an outward pointing unit vector normal to the boundary. The upper/lower sign refers to a Lorentzian/Euclidean world-sheet. The quantity χ is the *Euler number* of a worldsheet with boundary.

(e) Use the results (2.5) and (2.4) to argue that (locally) every metric of signature $(-1, 1)$ can be brought into the form $\eta_{\alpha\beta} = \text{diag}(-1, 1)$ by a Weyl rescaling and a diffeomorphism.

For questions:

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