String Theory I

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Exercise sheet 2

Due on **Thursday, October 31**, 2019
(Note that Friday, November 1, is a holiday)

Exercise 1: Longitudinal waves on non-relativistic strings

Consider a string with uniform mass density μ stretched between $x = 0$ and $x = a$ and with equilibrium tension T_0. Longitudinal waves are possible if the tension of the string varies as it stretches or compresses. For a piece of this string with equilibrium length L, a small change ΔL of its length is accompanied by a small change of the tension

$$\Delta T = \tau_0 \frac{\Delta L}{L} \ , \tag{1.1}$$

where τ_0 is a tension coefficient. Find the equation governing the small longitudinal oscillations of this string. Give the velocity of the waves.

Exercise 2: Induced metric on S^2 from stereographic parameterization

Consider a unit sphere S^2 in $\mathbb{R}^3$ centered at the origin: $x^2 + y^2 + z^2 = 1$. In the stereographic parameterization of the sphere one uses parameters ξ^1 and ξ^2 to parameterize a point on the sphere, i.e.

$$\vec{x}(\xi^1, \xi^2) = \Big(x(\xi^1, \xi^2), y(\xi^1, \xi^2), z(\xi^1, \xi^2) \Big) \ . \tag{2.1}$$

Given parameters (ξ^1, ξ^2), the corresponding point on the sphere lies on the line that goes through the north pole $N = (0, 0, 1)$ and the point $(\xi^1, \xi^2, 0)$ in the (x, y)-plain.

(a) Draw a sketch for the above construction. What are the required ranges for ξ^1 and ξ^2 if we wish to parametrize the full sphere?[1]

(b) Calculate the functions $x(\xi^1, \xi^2), y(\xi^1, \xi^2)$ and $z(\xi^1, \xi^2)$.

[1] Actually, the described parametrization is not well defined at the north pole itself. Thus, in this way one can only parametrize the sphere without the north pole. In order to find a coordinate system for the whole sphere using stereographic projection, one can cover the sphere with two patches, one excluding the north pole and one and excluding the south pole and then use the stereographic parametrization with the north and south pole as reference points, respectively.

(c) Calculate the four components of the induced metric

$$g_{ij}(\xi) = \frac{\partial \vec{x}}{\partial \xi^i} \cdot \frac{\partial \vec{x}}{\partial \xi^j} \ . \tag{2.2}$$

The algebra is a bit messy, but the result is simple. You can use a computer program like Maple or Mathematica for this part.

(d) Check your result by computing the area of the sphere using the formula from class:[2]

$$A = \int d\xi^1 d\xi^2 \sqrt{g} \ . \tag{2.3}$$

Exercise 3: Reparameterization invariance of the area

Show that the area formula (2.3) is form invariant under a change of variables

$$\xi^i \longrightarrow \tilde{\xi}^i = \tilde{\xi}^i(\xi^1, \xi^2) \ . \tag{3.1}$$

Exercise 4: Motion of string endpoints

Use the explicit form of

$$\Pi^\sigma_\mu = T \frac{\dot{X}^2 X'_\mu - (\dot{X} \cdot X')\dot{X}_\mu}{\sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}} \tag{4.1}$$

to calculate $\Pi^\sigma_\mu \Pi^{\sigma\mu}$. Use the result of this calculation to prove that free open string endpoints move with the speed of light.

For questions:

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[2]The fact that your calculations in part (b) and (c) are excluding the north pole is irrelevant for the present computation, because the you are neglecting just a single point in (2.3) and the metric is finite in the vicinity of that point.