String Theory I

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Exercise sheet 13

Due on January 31, 2020

Note: The exam will take place on Tuesday, February 25, at 9.30 am.

Exercise 1: Ghost CFT

The ghost CFT has the energy momentum tensor

$$T(z) = 2\!:\!\partial c(z) b(z)\!: + :\!c(z)\partial b(z)\!: \ . \tag{1.1}$$

(a) Calculate the central charge of the ghost system by considering the OPE of two energy momentum tensors. In order to perform the cross contractions you will need the OPEs

$$b(z_1)c(z_2) = c(z_1)b(z_2) = \frac{1}{z_1 - z_2} + \text{finite} \quad , \quad b(z_1)b(z_2) = \text{finite} \quad , \quad c(z_1)c(z_2) = \text{finite} \ . \tag{1.2}$$

(b) By considering the OPE of $T(z)$ with c and b, calculate the weights of c and b.
Hint: You have to be careful with the signs. In order to perform the contractions, you first have to commute the fields in such a way that they are next to each other.

Exercise 2: The moduli space of T^2

A torus can be represented as a rectangle in the plane with opposite sides identified, cf. figure 1. One can

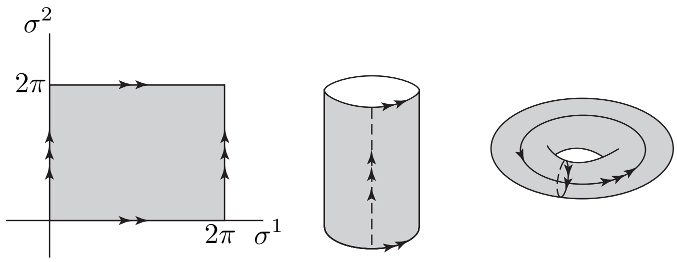

Figure 1: Torus

parametrize the torus with the coordinates σ^1 and σ^2 in the region

$$0 \leq \sigma^1 \leq 2\pi \ , \quad 0 \leq \sigma^2 \leq 2\pi \ , \tag{2.1}$$

with the metric $g_{ab}(\sigma^1, \sigma^2)$ periodic in both directions. Alternatively, one can think of this as the whole σ plane with the identification of points

$$(\sigma^1, \sigma^2) \cong (\sigma^1, \sigma^2) + 2\pi(m, n) \ , \quad m, n \in \mathbb{Z} \ . \tag{2.2}$$

We would now like to argue that one can bring a general metric on the torus into the form

$$ds^2 = |d\sigma^1 + \tau d\sigma^2|^2 \ , \quad \tau \in \mathbb{C} \ , \tag{2.3}$$

leaving the periodicity (2.1) intact.

To see this, one can first argue, similar to our earlier discussion, that one can make the metric flat by a Weyl transformation $g' = e^{2\omega} g$, where the Weyl factor fulfills $2\nabla^2\omega = R$, cf. ex. 2, sheet 3. This has a unique solution up to the addition of a constant to ω. For the torus, this can be done globally, as its Euler number vanishes and there is no obstruction to having a globally flat metric.

(a) By introducing new coordinates $\tilde{\sigma}^a$, one can bring the metric to unit form $g = \mathbf{1}_2$. Show that this changes the original periodicity to

$$\tilde{\sigma}^a \cong \tilde{\sigma}^a + 2\pi(mu^a + nv^a) \ , \tag{2.4}$$

with two linearly independent vectors u^a and v^a. Depending on the original flat metric, arbitrary (linearly independent) vectors u^a and v^a can arise.

By rotating the coordinates and rescaling them with an overall factor (accompanied by a Weyl-transformation in order to keep the components of the metric in unit form), one can set $u = (1, 0)$. This leaves the components of v as two parameters. Defining $w = \tilde{\sigma}^1 + i\tilde{\sigma}^2$, the metric is $dwd\bar{w}$ and the periodicity is

$$w \cong w + 2\pi(m + n\tau) \ , \quad \tau = v^1 + iv^2 \ , \tag{2.5}$$

cf. figure 2. Alternatively, one can define the coordinates σ^a by $w = \sigma^1 + \tau\sigma^2$. Obviously, these coordinates

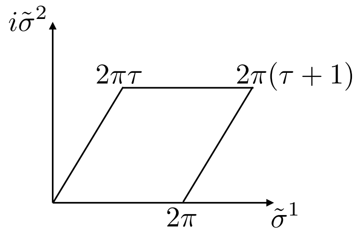

Figure 2: Coordinate region of $\tilde{\sigma}^a$.

have the original periodicity (2.1), but using these coordinates the metric takes the more general form (2.3).

Remark: The parameter τ is called a *modulus* or a *Teichmüller parameter*. The upshot of the above discussion is that one can parametrize inequivalent metrics on the torus in two different ways. One can either work with coordinates in which the metric takes the unit form and the modulus τ appears in the periodicity of the coordinates, cf. (2.5), or one can work with a fixed periodicity of the coordinates and then the modulus appears in the form of the metric, cf. (2.3). This second description is typically used in the path integral when integrating over inequivalent metrics.

(b) The metric (2.3) is obviously invariant under complex conjugation of τ and it is degenerate for real τ (show this). Thus, one can restrict attention to $\mathrm{Im}\,\tau > 0$. There is actually an additional redundancy. Consider the coordinate transformation

$$\begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix} \begin{pmatrix} \sigma'^1 \\ \sigma'^2 \end{pmatrix} \ , \quad a, b, c, d \in \mathbb{Z} \ , \quad ad - bc = 1 \ . \tag{2.6}$$

Show that this takes the metric (2.3) into a metric of the same form in σ' (up to a Weyl-rescaling) but with modulus

$$\tau' = \frac{a\tau + b}{c\tau + d} \ . \tag{2.7}$$

Argue that σ' has the same periodicity as σ, i.e. (2.1).

Remark: The coordinate transformations (2.6) are not continuously connected to the identity (with the exception of $a = d = 1, b = c = 0$). They are called *large coordinate transformations* and form the group $SL(2, \mathbb{Z})$, which is called the *modular group* of the torus. Using the modular transformations (2.7) every τ is equivalent to exactly one point in a *fundamental region*, which can be chosen for example as

$$\mathscr{F}_0 : \quad -\frac{1}{2} \leq \mathrm{Re}\,\tau \leq \frac{1}{2} \ , \quad |\tau| \geq 1 \ , \tag{2.8}$$

cf. figure 3, where the left (dashed) boundary has to be identified with the right boundary. This form of the fundamental region might seem plausible because the modular transformations (2.6) contain $\tau' = \tau + 1$ and $\tau' = -\frac{1}{\tau}$ as special cases. However, the actual proof is non-trivial and can be found for instance in chapter 26.6 of the book by Zwiebach. (To see that the statement is indeed non-trivial, note that the region $-\frac{1}{2} \leq \mathrm{Re}\,\tau \leq \frac{1}{2}$, $|\tau| \leq 1$ can *not* be chosen as a fundamental region.)

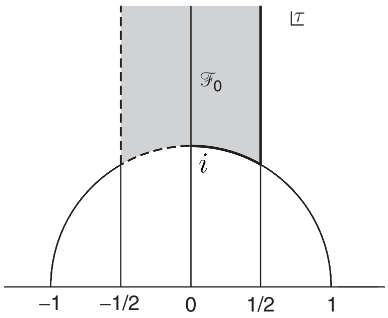

Figure 3: Fundamental region for the modulus τ of the torus.

Exercise 3: The 1-loop vacuum amplitude

The (1-loop) contribution to the vacuum energy in oriented closed string theory is given by the torus amplitude without any vertex operators. In order to discuss this amplitude, let us start by considering the vacuum energy in QFT, in particular for the field theory of a scalar with mass M in D dimensions described by the (Euclidean) action

$$S = \int d^D x \left(\tfrac{1}{2}\partial_\mu \phi \partial^\mu \phi + \tfrac{1}{2} M^2 \phi^2 \right) \ . \tag{3.1}$$

The vacuum energy Γ is defined via the path integral

$$e^{-\Gamma} = \int \mathcal{D}\phi \ e^{-S} \sim \det{}^{-1/2} \left(-\Delta + M^2 \right) \ . \tag{3.2}$$

This can be further rewritten by using the identity

$$\ln(\det(A)) = -\int_\epsilon^\infty \frac{dt}{t} \mathrm{tr} \left(e^{-tA} \right) \ , \tag{3.3}$$

where ϵ is an ultraviolet cutoff and t is a Schwinger parameter. The kinetic operator can be diagonalized by the complete set of momentum eigenstates, which results in

$$\Gamma = -\frac{V}{2} \int_\epsilon^\infty \frac{dt}{t} e^{-tM^2} \int \frac{d^D p}{(2\pi)^D} e^{-tp^2} \ , \tag{3.4}$$

where the space-time volume V arises from the continuum normalization of the momentum, i.e. Σ_p becomes $V(2\pi)^{-D} \int d^D p$ (see for example eq. (2.28) in Ashcroft, Mermin "Solid State Physics"). Performing the Gaussian momentum integrals yields

$$\Gamma = -\frac{V}{2(4\pi)^{D/2}} \int_\epsilon^\infty \frac{dt}{t^{D/2+1}} e^{-tM^2} \ . \tag{3.5}$$

For several bosonic fields, one has to sum over their contributions, i.e.

$$\Gamma_{\mathrm{tot}} = -\frac{V}{2(4\pi)^{D/2}} \int_\epsilon^\infty \frac{dt}{t^{D/2+1}} \mathrm{tr} \left(e^{-tM^2} \right) \ , \tag{3.6}$$

where the trace is over the entire mass spectrum.

One can now try to apply (3.6) to the bosonic string in $D = 26$, whose mass spectrum is encoded in

$$M^2 = \frac{2}{\alpha'} (N^\perp + \tilde{N}^\perp - 2) \ , \tag{3.7}$$

subject to the level matching condition $N^\perp = \tilde{N}^\perp$.

(a) Impose the level matching condition by using the integral representation of the Kronecker delta

$$\int_{-1/2}^{1/2} ds \ e^{2\pi i s (N^\perp - \tilde{N}^\perp)} = \delta_{N^\perp, \tilde{N}^\perp} \ . \tag{3.8}$$

Show that the result can be written as

$$\Gamma_{\text{tot}} = -\frac{V}{2(4\pi^2\alpha')^{13}} \int_{-1/2}^{1/2} d\tau_1 \int_{\epsilon}^{\infty} \frac{d\tau_2}{\tau_2^{14}} \, \mathrm{tr}\left(q^{N^\perp - 1} \bar{q}^{\tilde{N}^\perp - 1}\right) , \tag{3.9}$$

where we introduced the complex Schwinger parameter

$$\tau = \tau_1 + i\tau_2 = s + i\frac{t}{\alpha'\pi} \tag{3.10}$$

and the notation

$$q = e^{2\pi i\tau} \quad , \quad \bar{q} = e^{-2\pi i\bar{\tau}} \ . \tag{3.11}$$

This formula shows an ultraviolet divergence for $\epsilon \to 0$, as usual for field theory. In string theory, an exact treatment would lead to a similar formula as (3.9) with the τ of (3.10) given by the modulus of the world-sheet torus. However, the torus modulus is only integrated over the fundamental region $\mathscr{F}_0$, cf. (2.8), which introduces an effective ultraviolet cutoff!

(b) Up to an overall factor, the torus amplitude without vertex operators is, thus, given by ($d^2\tau \equiv d\tau d\bar{\tau}$)

$$\mathcal{T} = \int_{\mathscr{F}_0} \frac{d^2\tau}{\tau_2^2} \frac{1}{\tau_2^{12}} \mathrm{tr}\left(q^{N^\perp - 1} \bar{q}^{\tilde{N}^\perp - 1}\right) . \tag{3.12}$$

The trace over the bosonic string spectrum can be performed analogously to ex. 3 (a), sheet 6, resulting in

$$\mathcal{T} = \int_{\mathscr{F}_0} \frac{d^2\tau}{\tau_2^2} \frac{1}{\tau_2^{12}} \frac{1}{|\eta(\tau)|^{48}} , \tag{3.13}$$

with the *Dedekind* η function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \ . \tag{3.14}$$

Show that both, the measure $\tau_2^{-2} d^2\tau$ as well as the integrand $\tau_2^{-12}|\eta(\tau)|^{-48}$ of (3.13), are invariant under an $SL(2,\mathbb{Z})$ transformation (2.7). To do so, show first that

$$d^2\tau \to d^2\tau' = \frac{d^2\tau}{|c\tau + d|^4} \quad , \quad \tau_2 \to \tau_2' = \frac{\tau_2}{|c\tau + d|^2} \ . \tag{3.15}$$

Moreover, use (without proof) that a general $SL(2,\mathbb{Z})$ transformation (2.7) can be generated by a sequence of T and S transformations

$$T: \ \tau \to \tau' = \tau + 1 \ , \quad S: \ \tau \to \tau' = -\frac{1}{\tau} \ . \tag{3.16}$$

The transformation of the Dedekind η function under the generators T and S is

$$T: \ \eta(\tau + 1) = e^{\frac{i\pi}{12}} \eta(\tau) \quad , \quad S: \ \eta(-1/\tau) = \sqrt{-i\tau} \ \eta(\tau) \ . \tag{3.17}$$

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