String Theory I

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Exercise sheet 12

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Virasoro-Shapiro amplitude

The Virasoro-Shapiro amplitude gives the result of the scattering of four closed string tachyons at tree-level, i.e. the world-sheet takes the form of a sphere. In class we will obtain this amplitude using the path integral formalism. There is an alternative way to calculate it, the *operator formalism*, which should be employed in this exercise.

After gauge fixing the metric on the sphere (i.e. the plane with a point added at infinity), one still has some gauge symmetry left: those conformal transformations which are globally defined on the sphere. This left over gauge symmetry allows to fix the position of any three vertex operators to arbitrary positions, for instance $z_1 = 0$, $z_2 = 1$ and $z_3 = \infty$ (cf. ex. 2, sheet 10). Thus, we are interested in calculating

$$\mathcal{A}^{(4)} \sim \int d^2z \left\langle : e^{ik_3\cdot X(\infty,\infty)} :' \ : e^{ik_4\cdot X(z,\bar{z})} : \ : e^{ik_2\cdot X(1,1)} : \ : e^{ik_1\cdot X(0,0)} : \right\rangle \ , \tag{1.1}$$

where the prime at the first vertex operator indicates that it is the operator conformally mapped via $z_3 \to z_3' = 1/z_3$; by a slight abuse of notation we still give the position in terms of z_3. Moreover, the z-integral is over the whole Riemann sphere. Using the state-operator map, this can be rewritten as

$$\int d^2z \left\langle 0; k_3 \middle| R\Big(: e^{ik_4\cdot X(z,\bar{z})} : : e^{ik_2\cdot X(1,1)} : \Big) \middle| 0; k_1 \right\rangle \ , \tag{1.2}$$

where R denotes radial ordering as usual.

Normal ordering amounts to placing annihilation operators to the right of the creation operators, i.e.

$$: e^{ik\cdot X(z,\bar{z})} : \ = e^{ik\cdot X_C(z,\bar{z})} e^{ik\cdot X_A(z,\bar{z})} \ , \tag{1.3}$$

and it is conventional to group x^μ with the creation operators and p^μ with the annihilation operators, i.e.

$$\begin{aligned} X_C^\mu(z,\bar{z}) &= x^\mu - i\sqrt{\frac{\alpha'}{2}} \sum_{m=1}^{\infty} \frac{1}{m}(\alpha_{-m}^\mu z^m + \tilde{\alpha}_{-m}^\mu \bar{z}^m) \ , \\ X_A^\mu(z,\bar{z}) &= -i\frac{\alpha'}{2} p^\mu \ln|z|^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\alpha_m^\mu}{z^m} + \frac{\tilde{\alpha}_m^\mu}{\bar{z}^m}\right) \ . \end{aligned} \tag{1.4}$$

(a) Use the Baker-Campbell-Hausdorff formula, i.e. ($X_i \equiv X(z_i, \bar{z}_i)$)

$$e^{ik_1 \cdot X_{1A}} e^{ik_2 \cdot X_{2C}} = e^{ik_2 \cdot X_{2C}} e^{ik_1 \cdot X_{1A}} e^{-[k_1 \cdot X_{1A}, k_2 \cdot X_{2C}]} \ , \tag{1.5}$$

and the commutators

$$[\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0} \quad , \quad [\tilde{\alpha}_m^\mu, \alpha_n^\nu] = 0 \quad , \quad m, n \in \mathbb{Z} \ ,$$
$$[x^\mu, p^\nu] = i\eta^{\mu\nu} \quad , \quad [\alpha_n^\mu, x^\nu] = 0 \quad , \quad n \neq 0 \ , \tag{1.6}$$

to show for $|z_1| > |z_2|$:

$$e^{ik_1 \cdot X_{1A}} e^{ik_2 \cdot X_{2C}} = e^{ik_2 \cdot X_{2C}} e^{ik_1 \cdot X_{1A}} |z_1 - z_2|^{\alpha' k_1 \cdot k_2} \ . \tag{1.7}$$

(b) Use (1.3) and (1.7) to show that (1.2) is given by

$$\delta^D\Big(\sum_{i=1}^4 k_i\Big) \int d^2z |1-z|^{-\alpha' t/2 - 4} |z|^{-\alpha' u/2 - 4} \ , \tag{1.8}$$

up to an overall constant. Here, we used the Mandelstam variables

$$s = -(k_1 + k_2)^2 \quad , \quad t = -(k_1 + k_3)^2 \quad , \quad u = -(k_1 + k_4)^2 \ , \tag{1.9}$$

which obey

$$s + t + u = -\sum_i k_i^2 = \sum_i m_i^2 = -\frac{16}{\alpha'} \ . \tag{1.10}$$

Hint: Use $e^{ik' \cdot x}|0; k\rangle = |0; k + k'\rangle$ and $\langle 0; k|0; k'\rangle \sim \delta^D(k + k')$.

(c) Solve the integral (1.8) by showing

$$C(a, b) \equiv \int d^2z |z|^{2a-2} |1-z|^{2b-2} = 2\pi \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b)\Gamma(a+c)\Gamma(b+c)} \quad , \quad a + b + c = 1 \ . \tag{1.11}$$

Hint: Start by showing

$$|z|^{2a-2} = \frac{1}{\Gamma(1-a)} \int_0^\infty dt\ t^{-a} e^{-|z|^2 t} \tag{1.12}$$

and similarly for $|1 - z|^{2b-2}$. Use this in (1.11) and decompose the complex coordinate $z = x + iy$. Now first perform the integrals over x and y which are simply Gaussian. You should obtain

$$C(a, b) = \frac{2\pi}{\Gamma(1-a)\Gamma(1-b)} \int_0^\infty du dt \frac{t^{-a} u^{-b}}{t + u} e^{-tu/(t+u)} \ . \tag{1.13}$$

To make contact with (1.11) perform a change of variables $t = \alpha\beta$ and $u = (1 - \beta)\alpha$, with $\alpha \in [0, \infty)$ and $\beta \in [0, 1]$. You can then use the integral representation of the Euler beta function

$$B(x, y) \equiv \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} \ , \tag{1.14}$$

i.e.

$$B(x, y) = \int_0^1 dt\ t^{x-1}(1 - t)^{y-1} \ , \tag{1.15}$$

valid for $\text{Re}(x) > 0$ and $\text{Re}(y) > 0$.

(d) Using (1.11) in (1.8) you get the amplitude for the scattering of four closed string tachyons (up to an overall constant). The derivation is performed in a region of momentum space where the occurring integrals are convergent. The final result, however, is an analytic function of s, t and u, exhibiting poles at certain values. Via analytic continuation, it describes the scattering of four tachyons for arbitrary momenta. Where are the poles of the result?

(e) Discuss the result in the hard scattering limit, i.e. the limit of large center of mass energy and fixed (finite) angle. To do so, use that for a scattering process $1 + 2 \to 3 + 4$ of equal mass particles (of mass m), the Mandelstam variables are related to the center of mass energy E and the scattering angle θ (i.e. the angle between particles 1 and 3) as

$$s = E^2 \quad , \quad t = (4m^2 - E^2)\sin^2\frac{\theta}{2} \quad , \quad u = (4m^2 - E^2)\cos^2\frac{\theta}{2} \ . \tag{1.16}$$

What does the hard scattering limit imply for s, t and u? Use the approximation $\Gamma(x) \approx \exp(x \ln x)$ (valid for $|x| \to \infty$)[1] to show that

$$\mathcal{A}^{(4)} \approx \exp\Big[-\frac{\alpha'}{2}\Big(s\ln(s\alpha') + t\ln(t\alpha') + u\ln(u\alpha')\Big)\Big] \sim \exp\Big[-\frac{\alpha'}{2}sf(\theta)\Big] \ , \tag{1.17}$$

where in the second step a phase is neglected and $f(\theta)$ can be approximated as

$$f(\theta) \approx -\sin^2\frac{\theta}{2}\ln\Big(\sin^2\frac{\theta}{2}\Big) - \cos^2\frac{\theta}{2}\ln\Big(\cos^2\frac{\theta}{2}\Big) \ , \tag{1.18}$$

which is non-negative. Thus, the amplitude is exponentially suppressed in the hard scattering limit. This is a general feature of string scattering amplitudes (in particular also valid for the massless states). This soft high energy behavior is due to the extended nature of strings and is in contrast to the power law fall off found for the scattering of particles.

For questions:

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[1] Strictly speaking this formula does not hold along the negative real axis. Implicitly we move away from the real axis by adding a small imaginary part to s. This is ultimately justified because all the higher mass string states are actually unstable in the interacting theory and, thus, their poles are shifted away from the real axis. This can be mimicked by leaving the poles on the real axis, but instead taking the large s limit slightly away from the real axis.