String Theory I

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Exercise sheet 11

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Exercise 1: Correspondence between highest weight states and primary fields

Consider a local field $\mathcal{O}$ with scaling weights h and $\tilde{h}$, i.e. under a rescaling $z \to \lambda z$ (with $\lambda \in \mathbb{C}$) it transforms as $\mathcal{O} \to \lambda^{-h}\bar{\lambda}^{-\tilde{h}}\mathcal{O}$. Its OPE with $T(z)$ takes the form

$$T(z)\mathcal{O}(w,\bar{w}) = \text{more divergent} + \frac{h}{(z-w)^2}\mathcal{O}(w,\bar{w}) + \frac{1}{z-w}\partial\mathcal{O}(w,\bar{w}) + \text{finite} \tag{1.1}$$

(and similarly for $\tilde{T}(\bar{z})$). For a primary field, there are no more singular terms in the OPE than the ones given explicitly in (1.1), whereas for a secondary field there is at least one more singular term involving an operator which creates a non-null state from the vacuum via the state-operator correspondence. Consider the state $|\mathcal{O}\rangle = \mathcal{O}(0)|0\rangle$ and use $T = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}}$ in order to show

- $L_{-1}|\mathcal{O}\rangle = |\partial\mathcal{O}\rangle$,
- $L_0|\mathcal{O}\rangle = h|\mathcal{O}\rangle$,
- $L_n|\mathcal{O}\rangle = 0$ for all $n > 0$, if and only if $\mathcal{O}$ is a primary operator.

(Analogous results hold for $\tilde{L}_m$.)

Exercise 2: Unitary CFT's

Show that a unitary CFT (i.e. one without negative norm states) has the following two characteristics:

(i) $c \geq 0$, where c is the central charge,

(ii) $h \geq 0$, where h is the weight of any primary field (and similar for $\tilde{h}$).

Hint: You need to make use of the Virasoro algebra $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}$.

Remark: One can show that in a unitary CFT the only quasi primary field with $(h,\tilde{h}) = (0,0)$ is the unit operator (or multiples thereof) and the only state with L_0 and $\tilde{L}_0$ eigenvalues 0 is the $PSL(2,\mathbb{C})$-invariant vacuum state.

Exercise 3: Weights of some typical operators of the free X^μ CFT

(a) By computing the operator product expansions with $T(z) = -\frac{1}{\alpha'} :\partial X^\mu \partial X_\mu:$ and $\tilde{T}(\bar{z}) = -\frac{1}{\alpha'} :\bar{\partial} X^\mu \bar{\partial} X_\mu:$ and comparing with (1.1) (and similarly for $\tilde{T}(\bar{z})$), confirm that the following operators have the indicated scaling weights $(h, \tilde{h})$:

$$X^\mu \to (0,0) \ , \quad \partial X^\mu \to (1,0) \ , \quad \bar{\partial} X^\mu \to (0,1) \ , \tag{3.1}$$

$$\partial^2 X^\mu \to (2,0) \ , \quad :e^{ik\cdot X}: \to \left(\frac{\alpha' k^2}{4}, \frac{\alpha' k^2}{4} \right) \ . \tag{3.2}$$

Which of the operators are tensor operators?

(b) Calculate the singular terms in the OPE of $T(z_1)T(z_2)$.

Exercise 4: Closed string vertex operator

Consider the operator

$$\mathcal{O} = \zeta_{\mu\nu} : \partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X} : , \tag{4.1}$$

where $\zeta_{\mu\nu}$ is a constant tensor. Determine the condition for this operator to be primary of weight $(1,1)$, by looking at the operator product expansion with the energy momentum tensors $T = -\frac{1}{\alpha'} :\partial X_\rho \partial X^\rho:$ and $\tilde{T} = -\frac{1}{\alpha'} :\bar{\partial} X_\rho \bar{\partial} X^\rho:$.

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