String Theory I

Ludwig-Maximilians-Universität München
Dr. Michael Haack

Exercise sheet 10

Due on January 10, 2020

Exercise 1: The OPE of the energy momentum tensor

In the complex plane, the Virasoro generators L_n are given by

$$L_n = \oint_{C_0} \frac{dz}{2\pi i} z^{n+1} T(z) \ . \tag{1.1}$$

(a) Show that

$$[L_n, L_m] = \oint_{C_0} \frac{dz_2}{2\pi i} \oint_{C_{z_2}} \frac{dz_1}{2\pi i} z_1^{n+1} z_2^{m+1} T(z_1) T(z_2). \tag{1.2}$$

Here, C_0 denotes a contour around 0, C_{z_2} is a contour around z_2, and, as usual, the product $T(z_1)T(z_2)$ is meant to be the radially ordered product:

$$T(z_1)T(z_2) \equiv R(T(z_1)T(z_2)) = \left\{ \begin{array}{ll} T(z_1)T(z_2) & \text{for } |z_1| > |z_2| \\ T(z_2)T(z_1) & \text{for } |z_2| > |z_1| \end{array} \right\} . \tag{1.3}$$

(*Hint*: Write the commutator as a difference of two double contour integrals and use a contour deformation of the dz_1 integration for fixed z_2, just as was done in class for $\delta_{\epsilon Q}\phi(\tilde{z}) = \epsilon[Q, \phi(\tilde{z})]$.)

(b) Use (1.2) and the (radially ordered) operator product

$$T(z_1)T(z_2) = \frac{c/2}{(z_1 - z_2)^4} + \frac{2T(z_2)}{(z_1 - z_2)^2} + \frac{\partial_{z_2} T(z_2)}{(z_1 - z_2)} + \text{ (finite terms)} \ , \tag{1.4}$$

as well as Cauchy's integral formula,

$$\oint_{C_{z_2}} \frac{dz_1}{2\pi i} \frac{f(z_1)}{(z_1 - z_2)^n} = \frac{1}{(n-1)!} \partial^{(n-1)} f(z_2) \ , \tag{1.5}$$

to rederive the quantum Virasoro algebra:

$$[L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} n(n^2 - 1) \delta_{m+n,0} \ . \tag{1.6}$$

(c) Use (1.4) and the general formula for infinitesimal conformal transformations

$$\delta_{\epsilon\xi}\phi(z) = \epsilon[L_\xi, \phi(z)] \ , \tag{1.7}$$

where

$$L_\xi = -\oint_{C_0} \frac{dz}{2\pi i} \xi(z) T(z) \ , \tag{1.8}$$

to show that

$$\delta_{\epsilon\xi} T(z) = -\epsilon \Big[\frac{c}{12} \partial^3 \xi(z) + 2\partial\xi(z) T(z) + \xi(z) \partial T(z) \Big] \ . \tag{1.9}$$

(d) The Schwarzian derivative is defined as

$$\{f(z), z\} = \partial_z^3 f \big(\partial_z f\big)^{-1} - \frac{3}{2}\big(\partial_z^2 f\big)^2 \big(\partial_z f\big)^{-2} \ . \tag{1.10}$$

Show that the transformation

$$z \quad \to \quad z' \ , \tag{1.11}$$

$$T(z) \quad \to \quad T'(z') = \big(\partial_z z'\big)^{-2}\Big[T(z) - \frac{c}{12}\{z', z\}\Big] \tag{1.12}$$

gives (1.9) for an infinitesimal transformation $z' = z + \epsilon\xi(z)$.

Exercise 2: Fractional linear transformations

In this exercise we want to show that one can use conformal transformations on the Riemann-sphere (i.e. $\mathbb{C} \cup \{\infty\}$) to map any 3 points to any other 3 points. As will be discussed in more detail in class, the globally defined conformal transformations on the Riemann-sphere are given by the group $SL(2,\mathbb{C})$ of 2×2-matrices with unit determinant. They act on the points of the Riemann-sphere by fractional linear transformations

$$z \to z' = \frac{az+b}{cz+d} \quad , \quad \text{with} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) \ . \tag{2.1}$$

(a) Show that performing two successive fractional linear transformations

$$z \to z' = \frac{az+b}{cz+d} \quad , \quad z' \to z'' = \frac{ez'+f}{gz'+h} \tag{2.2}$$

is equivalent to performing the fractional linear transformation

$$z \to z'' = \frac{jz+k}{lz+m} \quad , \quad \text{with} \begin{pmatrix} j & k \\ l & m \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \tag{2.3}$$

Remark: This immediately implies that the inverse fractional transformation is given by the components of the inverse $SL(2,\mathbb{C})$-matrix.

(b) Consider the map

$$z \to z' = \frac{(x_2 - x_3)(z - x_1)}{(x_2 - x_1)(z - x_3)} \ . \tag{2.4}$$

Show that this defines an $SL(2,\mathbb{C})$ transformation if $x_1, x_2, x_3 \in \mathbb{C} \cup \{\infty\}$ are all distinct. Use this in order to show that one can map any 3 distinct points on the Riemann-sphere to any other 3 distinct points by an $SL(2,\mathbb{C})$ transformation.

(c) Show that the cross-ratio

$$\langle z_1, z_2, z_3, z_4 \rangle \equiv \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} \tag{2.5}$$

is invariant under an $SL(2,\mathbb{C})$ transformation.

For questions:
michael.haack@lmu.de