String Theory I

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Exercise sheet 1

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Exercise 1: Tensors and tensor densities

Consider a coordinate change

$$x^\mu \to x'^\mu \ . \tag{1.1}$$

Recall that tensors can be defined via their transformation properties under (1.1). More concretely, the components of say a $(1,1)$ tensor $T^\mu{}_\nu$ transform under (1.1) according to

$$T^\mu{}_\nu \to T'^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\nu} T^\rho{}_\sigma \ . \tag{1.2}$$

A $(1,1)$ tensor density $\tilde{T}^\mu{}_\nu$ of weight w is defined by the transformation property

$$\tilde{T}^\mu{}_\nu \to \tilde{T}'^\mu{}_\nu = |J|^w \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\nu} \tilde{T}^\rho{}_\sigma \ , \tag{1.3}$$

with $J = \det\left(\frac{\partial x^\mu}{\partial x'^\nu}\right)$ (and analogous definitions for tensor densities of different rank).

(a) Given a tensor $T_{\mu\nu}$, show that $(\det T_{\mu\nu})^{1/2}$ is a scalar density of weight 1.

(b) Consider now fields of tensors and tensor densities, e.g. $T^\mu{}_\nu(x)$ etc.. An infinitesimal transformation (1.1) can be expanded $x'^\mu = x^\mu - \epsilon^\mu(x) + \ldots$. Show the following infinitesimal transformations for a scalar field $\Phi(x)$, the metric $g_{\mu\nu}(x)$ and the associated density $(-\det g_{\mu\nu})^{1/2}$:

$$\begin{aligned}
\delta\Phi &= \epsilon^\mu \partial_\mu \Phi \ , & (1.4)\\
\delta g_{\mu\nu} &= \epsilon^\gamma \partial_\gamma g_{\mu\nu} + (\partial_\mu \epsilon^\gamma) g_{\gamma\nu} + (\partial_\nu \epsilon^\gamma) g_{\gamma\mu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu , & (1.5)\\
\delta(-\det g_{\mu\nu})^{1/2} &= \partial_\gamma(\epsilon^\gamma (-\det g_{\mu\nu})^{1/2}) \ . & (1.6)
\end{aligned}$$

The second equality of (1.5) assumes the metric compatible connection, i.e. $\nabla_\gamma g_{\mu\nu} = 0$. In order to verify that equality, use the Christoffel symbols $\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho})$.

Hint: A scalar field, for instance, is defined via $\Phi'(x') = \Phi(x)$ and $\delta\Phi$ in (1.4) is given by $\delta\Phi(x) \equiv \Phi'(x) - \Phi(x)$.

Exercise 2: Equation of motion for a point particle

(a) Consider the variation of the point particle action

$$S = \int_{\tau_i}^{\tau_f} d\tau \mathcal{L}\left(x^\mu(\tau), \frac{dx^\mu}{d\tau}(\tau)\right) \ , \tag{2.1}$$

under a variation $\delta x^\mu(\tau)$ of the particle trajectory. Assume that the variation δx vanishes at τ_i and τ_f. Show that Hamilton's action principle $\delta S = 0$ implies the Euler Lagrange equations

$$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \frac{\partial \mathcal{L}}{\partial x^\mu} \ , \tag{2.2}$$

where $\dot{x}^\mu = \frac{dx^\mu}{d\tau}(\tau)$.

(b) Specify now to the case of a charged particle moving in an electromagnetic field, albeit in flat space. Such a particle is described by the action

$$S = -m \int_{\mathcal{P}} ds + q \int_{\mathcal{P}} d\tau A_\mu(x(\tau)) \frac{dx^\mu}{d\tau}(\tau) \ , \tag{2.3}$$

where

$$ds^2 = -\eta_{\mu\nu}\, dx^\mu(\tau)\, dx^\nu(\tau) \ , \tag{2.4}$$

$\mathcal{P}$ is the worldline of the particle and the integral along it amounts to an integral from τ_i to τ_f when the worldline is parameterized by τ.

Use (2.2) to show that the equation of motion is

$$\frac{dp_\mu}{d\tau} = qF_{\mu\nu}\frac{dx^\nu}{d\tau} \ , \tag{2.5}$$

where

$$p_\mu = mu_\mu = m\frac{dx_\mu}{ds} \ , \tag{2.6}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{2.7}$$

(c) Now consider a neutral point particle moving in a curved space-time, i.e.

$$S = -m \int_{\mathcal{P}} ds \ , \quad ds^2 = -g_{\mu\nu}(x(\tau))\, dx^\mu(\tau)\, dx^\nu(\tau) \ . \tag{2.8}$$

Show that the equation of motion is the geodesic equation

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0 \ , \tag{2.9}$$

where the Christoffel symbols $\Gamma^\mu_{\nu\rho}$ were given at the end of exercise 1.

For questions:
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