LMU, Winter Term 2019/20

Exercises on Open Quantum Systems
Inés de Vega (Teacher)
Carlos Parra (Tutor)

Exercise 1 *Single particle physics of open quantum systems*

Let us consider a single two-level quantum system with frequency ω_s coupled to a bosonic environment, according to the Hamiltonian

$$H = \sum_{k=1}^{M} \omega_k b_k^\dagger b_k + \sum_{k=1}^{M} g_k (b_k \sigma^+ + b_k^\dagger \sigma^-) + \omega_s \sigma^+ \sigma^-, \tag{1}$$

where b_k ($b_k^\dagger$) are the bosonic annihilation (creation) operators for the mode k, and $\sigma^\pm$ are the spin ladder operators between the two atomic internal levels $|0\rangle, |1\rangle$. We now consider an initial state

$$|\Psi_0\rangle = |1\rangle|\{0\}\rangle, \tag{2}$$

i.e. where there is one excitation in the atom and the environment is in the vacuum state $|\{0\}\rangle$. Note that in the one excitation sector the Hamiltonian (1)

$$H = \sum_{k=1}^{M} \omega_k b_k^\dagger b_k + \sum_{k=1}^{M} g_k (|\{0\}\rangle\langle 1_k|\sigma^+ + |1_k\rangle\langle\{0\}|\sigma^-) + \omega_s \sigma^+ \sigma^-, \tag{3}$$

where $|1_k\rangle = |0_1, 0_2, \cdots, 1_k, \cdots, 0_M\rangle$ is the state with only one excitation in the mode k.

1. Since the Hamiltonian will preserve the number of excitations, the total system will have the general form

$$|\Psi(t)\rangle = C_0|0,0\rangle + A(t)|1,\{0\}\rangle + \sum_k B_k(t)|0,1_k\rangle, \tag{4}$$

   where $|1,\{0\}\rangle$ describes the excitation in the two-level system and no excitations in the environment, and $|0,1_k\rangle$ represents no excitations in the two-level systems and a single excitation in the bosonic mode k. By considering the time-dependent Schrödinger equation, assume that $B_k(0) = 0$, and show that

$$\frac{dA(t)}{dt} = -\int_0^t d\tau C(t-\tau)A(\tau), \tag{5}$$

   where $C(t) = \sum_\lambda g(k)^2 e^{-i\Delta_k t}$ is the correlation function of the environment, but with a displaced phase $\Delta_k = \omega_k - \omega_s$.

2. An analytical solution can be obtained using the Laplace transform method $A(t) = \mathcal{L}^{-1}(A(s)) = \mathcal{L}^{-1}K(s)$, where $K(s) = (\frac{A(0)}{s+C(s)})$, where $C(s)$ is the Laplace transform of the correlation function. The general solution of (5) is

$$A(t) = \int_{\epsilon - i\infty}^{\epsilon + i\infty} \mathrm{d}s K(s) \mathrm{e}^{st} = 2\pi \mathrm{i} \sum_j R_j - \int_C K(s) \mathrm{e}^{st}, \tag{6}$$

   where R_j are the residues of the function $K(s)e^{st}$ at the zeros of $s + C(s) = 0$ (which are the poles of $K(s)$). The function $K(s)$ has three different types of poles:

- Complex poles $s = -x + iy$
- Purely real poles $s = -x$
- Imaginary poles $s = iy$

To which physical scenarios will each solution correspond? Can you identify for which regime a bound state (i.e. entangled state between the system and the environment) will be formed in which the atomic population will not vanish completely at long times? Note that the last term of (6) corresponds to a contour integral around the poles. This term vanishes at long times and will not be important for our qualitative analysis we discuss above.

3. Write the form of the reduced density matrix in terms of $A(t)$, i.e. the form

$$\rho_s(t) = \begin{pmatrix} P(t) & D(t) \\ D^*(t) & 1 - P(t) \end{pmatrix}$$

where both $P(t)$ and $D(t)$ are functions of $A(t)$.

4. Write the form of the map ϕ_t by writing $\rho_s(t)$ as a vector $\rho_s^v(t)$ with four entries $(\rho_{00}, \rho_{01}, \rho_{10}, \rho_{11})$, such that you re-write the above equation

$$\rho_s^v(t) = \phi_t[\rho_s^v(0)]. \tag{7}$$

After this analytical inspection of the master equation, we now face the numerical resolution of the problem. To this aim, we consider the case of an atom in a cavity array. This means that in the Hamiltonian (1) we consider a periodic dispersion

$$\begin{aligned} \omega_k &= A + B\cos(kh), \\ g_k &= g, \end{aligned} \tag{8}$$

with $A = 100$, $B = 50$, $h = 1$, $g = 1$ as fixed quantities. In addition, we consider $M = 1000$ harmonic oscillators in the photonic environment. Correction: When generating the Hamiltonian for the photons, we should consider that the wave-vector runs from $-\pi$ to π as

$$k = \frac{2\pi j}{hM}, \tag{9}$$

with $j = -M/2 + 1, ..., M/2$. Note that if we want to consider two atoms (not necessary here) we should write that the second one is located at a distance L and would has an interaction Hamiltonian

$$H_I = g \sum_k (\sigma_2^\dagger b_k e^{ikL} + h.c), \tag{10}$$

where $k = j * dk$, where $j = 1, \cdots, M$.

Now, we consider the initial state (2) and address the following questions:

- Compute the time evolution of the quantum mean value of the observable $n_a(t) = \sigma^+(t)\sigma^-(t)$ for the following regimes:
  - $\omega_s = 51$,
  - $\omega_s = 49$.

What qualitative differences are observed in the dynamics? And if you place the frequency more inside the band, i.e. with $\omega_s = 53$?

- For the same cases, plot the real and the imaginary parts of $A(t)$, and relate the obtained result to the general structure (6). For which of values of ω_s we expect the presence of a bound state as described in the context of Eq. (6)?
- A bound state is an entangled state between the system and the environment that at long times has the general form

$$|\Psi(t)\rangle = ce^{iyt}|1, \{0\}\rangle + \sum_k igc\left(\frac{1 - e^{i(y+\omega_k-\omega_s)t}}{y + \omega_k - \omega_s}\right)|0, 1_k\rangle. \tag{11}$$

To check the bound state, compute the evolution of the von-Neumann entropy $S(\rho_s) = -\mathrm{Tr}_s\{\rho_s \mathrm{ln}\rho_s\}$ with time for the above cases $\omega_s = 51$ and $\omega_s = 49$ and determine when does the entropy persists. For total pure states such as the one we have, the von-Neumann entropy is an estimation of the system-environment entanglement.

5. Imagine now that we start from an initial state where the excitation is in the field, and not in the atom, i.e.

$$|\Psi_0\rangle = |0\rangle \sum_k B_k(0)|1_k\rangle, \tag{12}$$

where $\sum_k |B_k(0)|^2 = 1$ and $|1_k\rangle$ is a short notation for $|0, 0, \cdots, 1_k, 0, \cdots\rangle$, i.e the Fock state corresponding to a single excitation in the mode k. Note that there is still a single excitation but now it is spread in the electromagnetic field modes. Under which conditions will the excitation be best absorbed by the atom? How will the absorption probability depend on the initial distribution of $B_k(0)$?