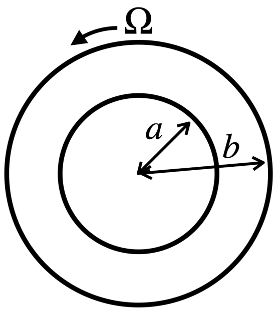
https://www2.physik.uni-muenchen.de/lehre/vorlesungen/sose_24/thermodynamik/index.html

Sheet 08

Discussion: Thursday 04.07.2024

Exercise 1 Navier-Stokes equation for cylindrical coordinates



Consider a slow, steady laminar flow in an annulus between an inner cylinder of radius a, which is at rest, and a concentric outer cylinder of radius b, which is rotating about its axis with angular velocity Ω (see figure above).

1. For slow flows, the Navier-Stokes equation simplifies: Argue, why in this case one of the terms in the Navier-Stokes equation can be neglected.

2. For the simplified Navier-Stokes equation, derive its form in cylindrical coordinates (r, φ, z). You should arrive at:

$$\rho\left[\frac{\partial v_r}{\partial t} - \frac{v_\varphi^2}{r}\right] = -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\varphi}{\partial \varphi}\right) \tag{1}$$

$$\rho\left[\frac{\partial v_\varphi}{\partial t} + \frac{v_\varphi v_r}{r}\right] = -\frac{1}{r}\frac{\partial p}{\partial \varphi} + \mu\left(\nabla^2 v_\varphi - \frac{v_\varphi}{r^2} - \frac{2}{r^2}\frac{\partial v_r}{\partial \varphi}\right) \tag{2}$$

$$\rho\frac{\partial v_z}{\partial t} = -\frac{\partial p}{\partial z} + \mu\nabla^2 v_z, \tag{3}$$

with

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}. \tag{4}$$

Hint: You can use the form of ∇f, $\nabla \times \vec{A}$ and $\nabla \cdot \vec{A}$ in cylindrical coordinates.

Assume the axial velocity u_z is everywhere zero as are all derivatives with respect to z. Furthermore, assume that $u_r = 0$ and that all derivatives with respect to φ are zero.

1. Calculate the velocity distribution $u_\varphi(r)$ within the annulus.

   Hint: Use the ansatz

$$x(r) = A/r + Br. \tag{5}$$

2. Find the difference in the pressures $p(b) - p(a)$ at the surfaces of the inner and outer cylinders.

3. Determine the shear stresses

$$\sigma|_{\mathrm{wall}} = \mu \frac{\mathrm{d}u_\varphi}{\mathrm{d}r}|_{\mathrm{wall}} \tag{6}$$

   acting on the surfaces of the inner and outer cylinders, and

4. the power $P = \sigma|_{r=b} A_{\mathrm{cylinder}} u_\varphi|_{r=b}$ required to rotate the outer cylinder.