https://www2.physik.uni-muenchen.de/lehre/vorlesungen/sose_24/t_m1_advanced-statistical-physics/index.html

Sheet 12:

Hand-out: Friday, Jul. 05, 2024; Hand-in: Sunday, Jul. 14, 2024, 11:59 pm

Problem 1 Scaling and fluids – (solution: Tutorials)

In the vicinity of the liquid-gas transition, the free energy is assumed to take the form

$$F/N = t^{2-\alpha} g(\delta\rho/t^{\beta}), \tag{1}$$

where $t = |T - T_c|/T_c$ is the reduced temperature and $\delta\rho = \rho - \rho_c$ is the deviation from the critical density. The leading order singular behavior in any thermodynamic parameter, $Q(t, \delta\rho)$, is of the form t^x when approaching along an isochore ($\rho = \rho_c$) or $\delta\rho^y$ when approaching along an isotherm ($T = T_c$). Assume a general form of Q, and keep in mind the singular behavior of $Q(0, \delta\rho)$ should not depend on t. Find the exponents x and y for the following:

(1.a) (**4 Points**) The internal energy per particle $\langle\mathcal{H}\rangle/N$ and the entropy per particle $s = S/N$.

(1.b) (**4 Points**) The heat capacity at constant volume, $C_V = T(\partial s/\partial T)_V$, and at constant pressure, $C_P = T(\partial s/\partial T)_P$.

(1.c) (**4 Points**) The isothermal compressibility, $\kappa_T = (\partial\rho/\partial P)_T/\rho$, and the thermal expansion coefficient $\alpha = (\partial V/\partial T)_P/V$.
Check that parts (b) and (c) are consistent with

$$C_P - C_V = TV\alpha^2/\kappa_T. \tag{2}$$

(1.d) (**4 Points**) Sketch the behavior of the latent heat per particle (L) on the coexistence curve for $T < T_c$, and find its singularity as a function of the reduced temperature t.

Problem 2 Position space RG in one-dimension – **(solution: Central Exercise)**

Consider the nearest neighbor Ising model in one dimension

$$\mathcal{H} = \sum_{i=1}^{N} B(\sigma_i, \sigma_{i+1}), \tag{3}$$

with the most general interaction between two neighboring spins, given by

$$B(\sigma_i, \sigma_{i+1}) = -g - \frac{h}{2}(\sigma_i + \sigma_{i+1}) - J\sigma_i\sigma_{i+1} = -\beta^{-1}\tilde{B}(\sigma_i, \sigma_{i+1}). \tag{4}$$

For this model an exact RG treatment can be carried out. The idea is to find a transformation that reduces the number of degrees of freedom by a factor b, while preserving the partition function, i.e.

$$Z = \sum_{\{\sigma_i\}_{i=1,\dots,N}} e^{-\beta\mathcal{H}(\{\sigma_i\})} = \sum_{\{\sigma'_{i'}\}_{i'=1,\dots,N/b}} e^{-\beta\mathcal{H}'(\{\sigma'_{i'}\})}. \tag{5}$$

There are many mappings $\{\sigma_i\} \to \{\sigma'_{i'}\}$ that satisfy this condition. The choice of the transformation is therefore guided by the simplicity of the resulting RG. Here we choose a so-called decimation procedure. The transformation is $\sigma_{i'} = \sigma_{2i-1}$, which removes the even numbered spins $s_i = \sigma_{2i}$.

Since the interaction is over adjacent neighbors, the partition function can be written as

$$Z = \sum_{\{\sigma_i\}_{i=1,\dots,N}} \exp\left[\sum_{i=1}^{N} \tilde{B}(\sigma_i, \sigma_{i+1})\right] =$$

$$= \sum_{\{\sigma'_{i'}\}_{i'=1,\dots,N/2}} \sum_{\{s_{i'}\}_{i'=1\dots N/2}} \exp\left[\sum_{i'=1}^{N/2} \left(\tilde{B}(\sigma'_{i'}, s_{i'}) + \tilde{B}(s_{i'}, \sigma'_{i'+1})\right)\right], \tag{6}$$

where the inverse temperature was absorbed into the interaction, $\tilde{B}(\sigma_i, \sigma_{i+1}) = -\beta B(\sigma_i, \sigma_{i+1})$.

Summing over the decimated spins, $\{s_i\}$, leads to

$$e^{-\beta\mathcal{H}'(\{\sigma'_{i'}\})} = \prod_{i'=1}^{N/2}\left[\sum_{s_{i'}=\pm 1} e^{B(\sigma'_{i'}, s_{i'}) + B(s_{i'}, \sigma'_{i'+1})}\right] \equiv e^{\sum_{i'=1}^{N/2} B'(\sigma'_{i'}, \sigma'_{i'+1})}. \tag{7}$$

(2.a) (**6 Points**) Show that the recursion relations for the interactions are

$$(z')^4 = z^8 \left(x^2 y + x^{-2} y^{-1}\right)\left(x^{-2} y + x^2 y^{-1}\right)\left(y + y^{-1}\right)^2, \tag{8}$$

$$(y')^2 = y^2 \frac{x^2 y + x^{-2} y^{-1}}{x^{-2} y + x^2 y^{-1}}, \tag{9}$$

$$(x')^4 = \frac{\left(x^2 y + x^{-2} y^{-1}\right)\left(x^{-2} y + x^2 y^{-1}\right))}{(y + y^{-1})^2}. \tag{10}$$

Here we have defined

$$x = e^{\beta J}, \qquad y = e^{\beta h}, \qquad z = e^{\beta g}, \tag{11}$$

and x', y', z' correspondingly.

(2.b) (**5 Points**) Consider the case with no external magnetic field ($h = 0$). Find the recursion relation for $K = \beta J$. Find the fixed points of the recursion relation, determine their stability and to which phases they correspond.

(2.c) (**5 Points**) Now the general case in the presence of a magnetic field. Consider the limiting cases $K = 0$, $K = \infty$, $h = 0$, $h = \pm\infty$ and sketch the flow diagram in (K, h) parameter space.

(2.d) (**5 Points**) By considering the recursion relations for K and h close to the fixed point at $K \to \infty$, $h = 0$, show that the correlation length ξ obeys the scaling form

$$\xi(e^{-K}, \beta h) = e^{2K} g_\xi(\beta h\, e^{2K}), \tag{12}$$

where $g_\xi(x)$ is an unknown function.

(2.e) (**5 Points**) Using the hyperscaling assumption for the singular part of the free energy

$$f_{\rm sing}(K, \beta h) \propto \xi^{-1}, \tag{13}$$

compare the behavior of the magnetic susceptibility $\chi \sim (\partial^2 f/\partial h^2)|_{h=0}$ and the correlation length ξ near zero temperature and at zero magnetic field. Then relate χ to the general form $\langle s_i s_{i+x}\rangle \sim e^{x/\xi}/x^{d-2+\eta}$ and compare the results to determine the critical exponent η.