Sheet 11:

Hand-out: Friday, Jun. 28, 2024; Hand-in: Sunday, Jul. 07, 2024, 11:59 pm

Problem 1 High temperature series for the Heisenberg model – **(solution: Central Exercise)**

Consider the spin-$1/2$ Heisenberg model on a simple cubic lattice, defined by the Hamiltonian

$$\hat{\mathcal{H}} = -J \sum_{\langle \boldsymbol{i},\boldsymbol{j} \rangle} \hat{\boldsymbol{S}}_{\boldsymbol{i}} \cdot \hat{\boldsymbol{S}}_{\boldsymbol{j}} - h \sum_{\boldsymbol{j}} \hat{S}^z_{\boldsymbol{j}}. \tag{1}$$

(1.a) (**4 Points**) Find the high-temperature series expansion for the susceptibility per spin

$$\chi(0,T) = \frac{\partial}{\partial h} \langle \hat{S}^z_{\boldsymbol{j}} \rangle |_{h=0} = \beta \frac{\mathrm{tr} \sum_{\boldsymbol{i}} \hat{S}^z_{\boldsymbol{i}} \hat{S}^z_{\boldsymbol{j}} \exp\left(\beta J \sum_{\langle \boldsymbol{r},\boldsymbol{x} \rangle} \hat{\boldsymbol{S}}_{\boldsymbol{x}} \cdot \hat{\boldsymbol{S}}_{\boldsymbol{r}}\right)}{\mathrm{tr} \exp\left(\beta J \sum_{\langle \boldsymbol{r},\boldsymbol{x} \rangle} \hat{\boldsymbol{S}}_{\boldsymbol{x}} \cdot \hat{\boldsymbol{S}}_{\boldsymbol{r}}\right)}, \tag{2}$$

up to and including terms of $\mathcal{O}(J^2)$.

(1.b) (**4 Points**) Analyze this short series by writing

$$\chi(0,T) = \frac{1}{T} \left(\frac{a_1 - a_2/T}{1 - a_3/T} \right), \tag{3}$$

which is a simple Padé approximation. Find T_c.

(1.c) (**2 Points**) Compare the critical temperature T_c from the two-term expansion with the best estimate $k_B T_c / J\hbar^2 = 0.84$ obtained from longer series.

Problem 2 Capillary waves – (solution: Tutorials)

A reasonably flat surface in d-dimensions can be described by its height h, as a function of the remaining $(d-1)$ coordinates $\boldsymbol{x} = (x_1, ..., x_{d-1})$. Convince yourself that the generalized area is given by $A = \int d^{d-1}x \ \sqrt{1 + (\nabla h(\boldsymbol{x}))^2}$. With a surface tension σ, the Hamiltonian is $\mathcal{H} = \sigma A$.

(2.a) (**3 Points**) At sufficiently low temperature, there are only slow variations in h. Expand the energy up to quadratic order, and write down the partition function Z as a functional integral.

(2.b) (**3 Points**) Use a Fourier transformation to diagonalize the quadratic Hamiltonian into its normal modes $\{h_{\boldsymbol{q}}\}$ (i.e. capillary waves).

(2.c) (**3 Points**) What symmetry breaking is responsible for these Goldstone modes?

(2.d) (**4 Points**) Calculate the height-height correlations $\langle (h(\boldsymbol{x}) - h(\boldsymbol{x}'))^2 \rangle$.

(2.e) (**4 Points**) Compare the asymptotic behavior $|\boldsymbol{x} - \boldsymbol{x}'| \to \infty$ of the height-height correlations in dimensions $d > 3$ and $1 < d \leq 3$.