https://www2.physik.uni-muenchen.de/lehre/vorlesungen/sose_24/t_m1_advanced-statistical-physics/index.html

Sheet 9:

Hand-out: Friday, Jun. 14, 2024; Hand-in: Sunday, Jun. 23, 2024, 11:59 pm

Problem 1 Gauge fluctuations in superconductors – **(solution: Central Exercise)**

The Ginzburg-Landau model of superconductivity describes a complex superconducting order parameter $\Psi(\boldsymbol{x}) = \Psi_1(\boldsymbol{x}) + i\Psi_2(\boldsymbol{x})$ and the electromagnetic vector potential $\boldsymbol{A}(\boldsymbol{x})$ which are subject to a Hamiltonian

$$\beta\mathcal{H} = \int d^3\boldsymbol{x} \left[\frac{t}{2}|\Psi|^2 + u|\Psi|^4 + \frac{K}{2}\left(D_\mu\Psi\right)\left(D_\mu^*\Psi^*\right) + \frac{L}{2}\left(\nabla\times\boldsymbol{A}\right)^2\right]. \tag{1}$$

The gauge-invariant derivative $D_\mu = \partial_\mu - ieA_\mu(\boldsymbol{x})$ introduces the coupling between the two fields. Suppose $u > 0$. Note that Einstein's summation convention is used.

(1.a) (**3 Points**) Show that the above Hamiltonian is invariant under the local gauge symmetry $\Psi(\boldsymbol{x}) \to \Psi(\boldsymbol{x})\exp(i\theta(\boldsymbol{x}))$ and $A_\mu(\boldsymbol{x}) \to A_\mu(\boldsymbol{x}) + \frac{1}{e}\partial_\mu\theta$.

(1.b) (**4 Points**) Show that there is a saddle point solution of the form $\Psi(\boldsymbol{x}) = \overline{\Psi}$ and $A(\boldsymbol{x}) = 0$, and find $\overline{\Psi}$ for $t > 0$ and $t < 0$.

(1.c) (**4 Points**) For $t < 0$, calculate the cost of fluctuations by setting $\Psi(\boldsymbol{x}) = \left(\overline{\Psi} + \phi(\boldsymbol{x})\right)\exp(i\theta(\boldsymbol{x}))$ and $A_\mu(\boldsymbol{x}) = a_\mu(\boldsymbol{x})$ (with $\partial_\mu a_\mu = 0$ in the Coulomb gauge) and expanding $\beta\mathcal{H}$ to quadratic order in ϕ, θ and $\boldsymbol{a}$. (ϕ, θ and a_μ are real fields).

(1.d) (**4 Points**) Perform a Fourier transformation on these newly introduced fields and calculate the expectation values of the fluctuations: $\langle|\phi(\boldsymbol{q})|^2\rangle$, $\langle|\theta(\boldsymbol{q})|^2\rangle$ and $\langle|\boldsymbol{a}(\boldsymbol{q})|^2\rangle$.

Problem 2 Müller-Hartmann and Zittartz estimate – (solution: Tutorials)

We consider an anisotropic 2D Ising model with interactions K_x (K_y) on bonds in the x-(y-)direction, on a $(L+1)\times(H+1)$ rectangular lattice with periodic boundary conditions in the x-direction, i.e. $x = L+1 \equiv 1$. It can be described by the Hamiltonian

$$\mathcal{H}_0 = -K_x\sum_{x=1}^{L}\sum_{y=1}^{H} S_{x,y}S_{x+1,y} - K_y\sum_{x=1}^{L}\sum_{y=1}^{H-1} S_{x,y}S_{x,y+1}, \tag{2}$$

and we assume $K_x, K_y > 0$.

(2.a) (**4 Points**) Consider an interface, where all configurations above (below) the interface are of type spin up (down). Ignoring islands and overhangs, the configurations can be labelled by heights h_n for $1 \leq n \leq L$ and $1 \leq h_n \leq H$. Show that the energy of an interface along the x-direction relative to the completely polarized state (ignoring edge effects) is

$$\mathcal{H} = 2K_y L + 2K_x \sum_{n=1}^{L} |h_{n+1} - h_n|. \tag{3}$$

This defines an effective one-dimensional (1D) model.

(2.b) (**4 Points**) Give an expression for the transfer matrix $\langle h|T|h'\rangle$ for the effective 1D model.

(2.c) (**4 Points**) In the limit that $H \to \infty$, obtain the free energy of the interface F_{int} via a direct summation of the partition function (without use of the transfer matrix result).

(2.d) (**4 Points**) Find K_x, K_y for which $F_{\text{int}} = 0$ for $H \to \infty$. An exact solution of the full model (3) gives the condition $\coth(\beta K_x) = e^{2\beta K_y}$. What can you conclude about the importance of unconnected islands of polarization?