Sheet 7:

Hand-out: Friday, May. 31, 2024; Hand-in: Sunday, Jun. 09, 2024, 11:59 pm

Problem 1 Infinite-range Ising model and Hubbard-Stratonovich – **(solution: Central Exercise)**

This question concerns the infinite-range Ising model, where the coupling constant $J_{ij} = J_0$ for all i, j (i.e. no restriction to nearest neighbor interactions). Consider

$$\mathcal{H}\{s\} = -\frac{J_0}{2}\sum_{i,j} s_i s_j - H\sum_j s_j, \qquad J_0 > 0. \tag{1}$$

You will solve this model using a method referred to as the Hubbard-Stratonovich transformation, or auxiliary fields. Although it is nothing more than completing the square, this technique is one of the most useful tricks in the physicist's arsenal. The virtue of the present example is that you will be able to calculate a partition function in an essentially exact way, and see precisely how it is that the thermodynamic limit or the zero temperature limit are required in order for there to be a phase transition.

(1.a) (**3 Points**) Explain why this model only makes sense if $J_0 = J/N$, where N is the number of spins in the system.

(1.b) (**5 Points**) Use the identity

$$\exp\left[\frac{a}{2N}x^2\right] = \int_{-\infty}^{\infty} dy\, \sqrt{\frac{Na}{2\pi}}\; e^{-\frac{Na}{2}y^2 + axy}, \tag{2}$$

with $\mathrm{Re}(a) > 0$, to show that the partition function Z is given by

$$Z = \int_{-\infty}^{\infty} dy\, \sqrt{\frac{N\beta J}{2\pi}}\; e^{-N\beta L(y)}, \tag{3}$$

where

$$L(y) = \frac{J}{2}y^2 - \frac{1}{\beta}\log\left[2\cosh\left(\beta(H + Jy)\right)\right]. \tag{4}$$

When can this expression become non-analytical?

(1.c) (**4 Points**) In the thermodynamic limit, this integral can be evaluated exactly by the method of steepest descents (also known as the saddle-point approximation). Show that

$$Z(\beta, H, J) = \sum_j e^{-\beta N L(H,J,\beta,y_j)} \tag{5}$$

and find the equation satisfied by the different saddle point solutions y_j. What is the probability of the system being in the state specified by y_j? Hence show the magnetization is given by

$$M \equiv \lim_{N\to\infty} \frac{1}{\beta N} \frac{\partial \log Z}{\partial H} = y_0, \tag{6}$$

where y_0 is the position of the global minimum of L.

(1.d) (**3 Points**) Now consider the case $H = 0$. By considering how to solve the equation for y_j graphically, show that there is a phase transition and find the transition temperature T_c. Discuss the acceptability of all the solutions of the equation for y_j both above and below T_c.

(1.e) (**3 Points**) Calculate the isothermal susceptibility

$$\chi_T \equiv \frac{\partial M}{\partial H}. \tag{7}$$

For $H = 0$, show that χ_T diverges to infinity both above and below T_c, and find the leading behavior of χ_T in terms of the reduced temperature $t = (T - T_c)/T_c$.

Problem 2 A phase transition near the Lifshitz point – (solution: Tutorials)

In this problem, we consider a class of magnetic materials which exhibit so-called *modulated phases*. For simplicity, we consider the case of a system of length L in one spatial dimension, and use the Landau free energy density:

$$F_{\text{Landau}} = \frac{1}{L}\int_0^L dx \, \left(\frac{a}{2}M^2 + \frac{b}{4}M^4 + \frac{\alpha}{2}\left(\frac{\partial M}{\partial x}\right)^2 + \frac{\beta}{4}\left(\frac{\partial^2 M}{\partial x^2}\right)^2 \right). \tag{8}$$

Here, $b > 0$, $\beta > 0$, and a and α may be of either sign. Below you will work out the phase diagram in the $\alpha - a$ plane.

(2.a) (**2 Points**) Define the Fourier transform

$$M(x) = \sum_{n=-\infty}^{\infty} e^{iq_n x} M_n, \qquad q_n = \frac{2\pi}{L} n. \tag{9}$$

Write out the inverse transform for M_n in terms of $M(x)$, and verify that you obtain the correct expression for the Kronecker and the Dirac delta functions.

(2.b) (**3 Points**) Write out the Landau free energy in terms of the Fourier components M_n.

(2.c) (**4 Points**) By minimizing with respect to both n and M_n, show that the system exhibits three possible phases: a paramagnetic phase ($M = 0$), a ferromagnetic phase ($M \neq 0$) and a spatial modulated phase ($M_q \neq 0$ for some $q \neq 0$).

(2.d) (**4 Points**) What is the wavelength of the modulation? Find the phase boundaries and draw the phase diagram. What is the order of the different phase transitions? You should assume that in the modulated phase, you only need to consider one Fourier component.