Sheet 6:

Hand-out: Friday, May. 24, 2024; Hand-in: Sunday, Jun. 02, 2024, 11:59 pm

Problem 1 The Potts model – **(solution: Central Exercise)**

An example that leads to a Landau expansion with a cubic term, which breaks inversion symmetry and gives rise to a a first order transition, is the Potts model. Consider a system of N spins, each of which can be in any of p states. Each spin only interacts with the z nearest neighbor spins of the same type as itself:

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \delta_{S_i,S_j}, \qquad J > 0. \tag{1}$$

For $p = 2$ this is just the Ising model. We consider the case $p = 3$ and label the allowed states $S_j \in \{A, B, C\}$. Let $n_A = N_A/N$, $n_B = N_B/N$ and $n_C = N_C/N$.

(1.a) (**4 Points**) Show that the free energy in the Bragg-Williams mean-field theory becomes:

$$F = -\frac{z}{2} NJ \left(n_A^2 + n_B^2 + n_C^2\right) + Nk_BT \left(n_A \ln n_A + n_B \ln n_B + n_C \ln n_C\right). \tag{2}$$

(1.b) (**2 Points**) In the disordered high-temperature phase $n_A = n_B = n_C = 1/3$. In general, these concentrations are subject to the constraint $n_A + n_B + n_C = 1$. A possible parametrization that anticipates a meaningful definition of an order parameter is

$$n_A = \frac{1}{3}(1 + 2y), \qquad n_B = \frac{1}{3}\left(1 + \sqrt{3}x - y\right), \qquad n_C = \frac{1}{3}\left(1 - \sqrt{3}x - y\right). \tag{3}$$

Graphically represent the allowed values of x and y.

(1.c) (**3 Points**) The possible ordered phases have preferential occupation of either the A, B or C state. Because of symmetry we can restrict ourselves to the A state ($x = 0$) and choose as order parameter $m = y$, $-1/2 \leq m \leq 1$. Show that around T_c, the free energy can be expanded as

$$F/N = -zJ/6 - k_BT \ln(3) - (zJ/3 - k_BT)m^2 - \frac{k_BT}{3}m^3 + \frac{k_BT}{2}m^4 + \mathcal{O}(m^5). \tag{4}$$

(1.d) (**3 Points**) Plot the exact (no expansion) free energy $F/(NzJ)$ versus m for different temperatures around $k_BT_c/(zJ) = 1/(4\ln(2)) \simeq 0.3607$. Determine the transition temperature T_c and the value of m at T_c numerically. Using the numeric value of m, you can confirm the exact result for T_c analytically.

(1.e) (**3 Points**) Since the transition is first order, it is accompanied by latent heat L. Show, by computing the entropy per spin in both phases, that $L = zJ/12$.

Problem 2 Transfer matrix solution of the 1D Potts model – (solution: Tutorials)

Consider the three-state Potts model of N spins in one dimension with periodic boundary conditions, defined by the Hamiltonian

$$\mathcal{H} = \sum_{j=1}^{N} \left(J\delta_{S_j,S_{j+1}} + \sum_{\alpha=1}^{3} h_\alpha \delta_{S_j,\alpha} \right), \tag{5}$$

with spin values $S_j \in \{1, 2, 3\}$. Use of some computer algebra system is suggested for this exercise.

(2.a) (**4 Points**) Write the partition function Z in terms of a transfer matrix $T \in \mathbb{R}^{3\times3}$.

(2.b) (**4 Points**) Let $h_2 = h_3 = 0$. Calculate the free energy per spin in the limit $N \to \infty$.

(2.c) (**4 Points**) Give an expression for a magnetization-like order parameter in analogy to the Ising model in the thermodynamic limit, and plot it as a function of $h_1\beta$ for $h_2 = h_3 = 0$, $J\beta = \{0.1, 1, 4\}$.