LUDWIG-
MAXIMILIANS-
UNIVERSITÄT
MÜNCHEN

Fakultät für Physik im SoSe 2024
T M1/TV: Advanced Statistical Physics
Dozent: Prof. Dr. Fabian Grusdt
Übungen: P. Bermes, P. Kopold, M. Moradi, F. Pauw, H. Riahi

https://www2.physik.uni-muenchen.de/lehre/vorlesungen/sose_24/t_m1_advanced-statistical-physics/index.html

Sheet 4:

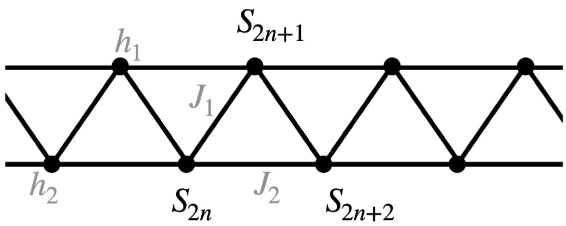
Hand-out: Friday, May. 10, 2024; Hand-in: Sunday, May. 19, 2024, 11:59 pm

Problem 1 An extended Ising model – **(solution: Central Exercise)**

Consider a triangular chain of Ising spins, $S_j = \pm 1$, with sub-lattice magnetic fields $h_{1,2}$ and nearest- plus next-nearest neighbor couplings $J_{1,2}$ respectively, described by the Hamiltonian:

$$\mathcal{H} = -\sum_j \left(J_1 S_{j+1} S_j + J_2 S_{j+2} S_j\right) + \sum_n \left(h_1 S_{2n+1} + h_2 S_{2n}\right). \tag{1}$$

The configuration we consider is sketched in the figure below, edge effects can be ignored throughout. The temperature of the system T can be parametrized by $\beta = 1/k_B T$.



(1.a) (**2 Points**) Write down a formal expression for the canonical partition function Z (Note: you do not need to evaluate any sums!).

(1.b) (**4 Points**) How large is the general transfer matrix for this model? Write down the transfer matrix $\hat{T}$ for the special case without fields, i.e. $h_1 = h_2 = 0$, but general $J_1, J_2 \neq 0$.

(1.c) (**3 Points**) Now consider the simpler case where $J_2 = 0$ but $h_1, h_2 \neq 0$. Explain why the transfer matrix $\hat{T} = \hat{T}_{\text{even}} \hat{T}_{\text{odd}}$ can be written as a product of two transfer matrices $\hat{T}_{\text{even}}$ and $\hat{T}_{\text{odd}}$ in this case. Show that they are:

$$\hat{T}_{\text{even}} = \begin{pmatrix} e^{-\beta(h_1+h_2)/2+\beta J_1} & e^{\beta(h_1-h_2)/2-\beta J_1} \\ e^{-\beta(h_1-h_2)/2-\beta J_1} & e^{\beta(h_1+h_2)/2+\beta J_1} \end{pmatrix}, \qquad \hat{T}_{\text{odd}} = \left(\hat{T}_{\text{even}}\right)^T. \tag{2}$$

Hint: You don't need results from (1.b) here!

(1.d) (**3 Points**) Use the expressions from (1.c) to derive a closed analytical expression for the thermodynamic free energy F/L in the case $J_1 = J_2 = 0$, where $L \to \infty$ is the length of the chain. To simplify notations, introduce $h_\pm = (h_1 \pm h_2)/2$.
Hint: A real symmetric 2×2 matrix $\hat{M}$ can be decomposed into a sum of Pauli matrices $\hat{\sigma}^\mu$ in the following way: $\hat{M} = m_0 + \sum_{\mu=x,y,z} m_\mu \hat{\sigma}^\mu$ with real numbers m_0, m_x, m_z and $m_y = 0$. This leads to the eigenvalues of $\hat{M}$ given by $\lambda_\pm = m_0 \pm \sqrt{\sum_\mu m_\mu^2}$.

Problem 2 Wick's / Isserlis' Theorem – (solution: Tutorials)

The goal of this problem is to prove the following important relation, valid for an important class of partition functions with an action quadratic in some classical continuous variables x_j; Namely, higher-order correlation functions can be decomposed into pair-wise correlations:

$$\langle x_{i_1}...x_{i_n}\rangle = \sum_I \langle x_{j_1}x_{k_1}\rangle...\langle x_{j_{n/2}}x_{k_{n/2}}\rangle \tag{3}$$

where the sum is over all possible pairings I (or, in the language of QFT, contractions) of $i_1, ..., i_n$ into pairs $(j_1, k_1), ..., (j_{n/2}, k_{n/2})$. E.g. for $n = 4$ one would get:

$$\langle x_{i_1}...x_{i_4}\rangle = \langle x_1x_2\rangle\langle x_3x_4\rangle + \langle x_1x_3\rangle\langle x_2x_4\rangle + \langle x_1x_4\rangle\langle x_2x_3\rangle. \tag{4}$$

(2.a) (**3 Points**) Start from a general partition function with quadratic action (characterized by a symmetric, positive definite matrix H) and a source term $\boldsymbol{J}$,

$$Z[\boldsymbol{J}] = \int d^n\boldsymbol{x}\ \exp\left[-\frac{1}{2}\sum_{ij} H_{ij}x_ix_j + \sum_i J_ix_i\right], \tag{5}$$

where we dropped constants and took $d^n\boldsymbol{x}$ to assume the appropriate multi-dimensional form. We will also assume normalization, i.e. $Z[0] = 1$ and $\boldsymbol{x} \in \mathbb{R}^n$ such that $\sum_{ij} H_{ij}x_ix_j = \boldsymbol{x}\cdot H\boldsymbol{x}$. With the definition

$$\langle x_qx_r\rangle = \int d^n\boldsymbol{x}\ x_q\ x_r\ \exp\left[-\frac{1}{2}\sum_{ij} H_{ij}x_ix_j\right], \tag{6}$$

show that

$$\left\langle \prod_i x_i^{a_i}\right\rangle = \prod_i \left(\frac{\partial^{a_i}}{\partial J_i^{a_i}}\right) Z[\boldsymbol{J}]\,\Bigg|_{\boldsymbol{J}=0}, \tag{7}$$

where $x_i^{a_i}$ denotes the i-th component of $\boldsymbol{x}$ to the a_i-th power ($a_i = 0$ is allowed).

(2.b) (**3 Points**) Next, expand $(H\boldsymbol{x} - \boldsymbol{J})\ H^{-1}(H\boldsymbol{x} - \boldsymbol{J})$ and rewrite the resulting equation into an expression for the exponent of Eq. (5). Use a variable transformation to conclude

$$Z[\boldsymbol{J}] = \exp\left[\frac{1}{2}\sum_{ij}(H^{-1})_{ij}J_iJ_j\right] Z[0] = \exp\left[\frac{1}{2}\sum_{ij}(H^{-1})_{ij}J_iJ_j\right], \tag{8}$$

and prove that $\langle x_ix_j\rangle = (H^{-1})_{ij}$.

(2.c) (**4 Points**) Show that Eq. (7) implies that $Z[\boldsymbol{J}]$ can be written in the form

$$Z[\boldsymbol{J}] = \sum_{n=0}^{\infty}\frac{1}{n!}\left[\sum_{i_1,...,i_n}\left\langle\prod_{k=1}^{n} x_{i_k}\right\rangle \prod_{k=1}^{n} J_{i_k}\right] =$$
$$= 1 + \sum_i \langle x_i\rangle J_i + \frac{1}{2}\sum_{ij}\langle x_ix_j\rangle J_1J_j + \frac{1}{6}\sum_{ijk}\langle x_ix_jx_k\rangle J_iJ_jJ_k + ... \tag{9}$$

Compare the coefficients of $J_aJ_b, J_aJ_bJ_cJ_d, ...$ in (9) and (8) to prove Eq. (3).

Problem 3 Phase transitions in 1D; Perron-Frobenius Theorem – **(solution: Central Exercise)**

In this exercise, you may assume the following theorem to hold without proof:

For the eigenvalue of largest magnitude λ of a real $m \times m$ matrix A with all matrix elements strictly positive ($A_{ij} > 0$), the following statements are true:

(i) $\lambda \in \mathbb{R}_{>0}$.

(ii) The eigenspace associated with λ has dimension 1, i.e. λ is a non-degenerate eigenvalue.

(iii) λ is an analytic function of all A_{ij}.

Now consider for concreteness a general spin system.

(3.a) (**3 Points**) How is the largest eigenvalue of the transfer matrix of such a system related to phase transitions in the thermodynamic limit?

(3.b) (**3 Points**) Assume the system to be one-dimensional. Which other restriction is necessary to guarantee a finite size $m \times m$ of the transfer matrix in the thermodynamic limit? What is the general form of the matrix elements of the transfer matrix?

(3.c) (**3 Points**) Assume the system to be one-dimensional and the transfer matrix to have finite size $m \times m$. When is the above theorem applicable to this matrix?

(3.d) (**3 Points**) Assume the system to be two-dimensional. Why does the above theorem *not* apply in this case in the thermodynamic limit in both directions? How is this inapplicability related to the one discussed in (3.b)?