LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

Fakultät für Physik im SoSe 2024
T M1/TV: Advanced Statistical Physics
Dozent: Prof. Dr. Fabian Grusdt
Übungen: P. Bermes, P. Kopold, M. Moradi, F. Pauw, H. Riahi

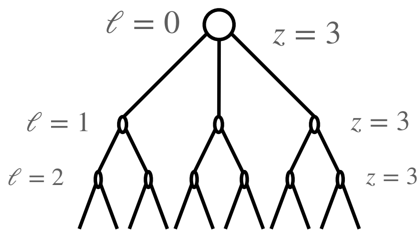
https://www2.physik.uni-muenchen.de/lehre/vorlesungen/sose_24/t_m1_advanced-statistical-physics/index.html

Sheet 1:

Hand-out: Friday, Apr. 19, 2024; Hand-in: Sunday, Apr. 28, 2024, 11:59 pm

Problem 1 Energy vs. Entropy – **(solution: Central Exercise)**

Consider a classical particle on a Bethe lattice with coordination number z, i.e. each node has z nearest neighbors. Assume further that the energy of the particle vanishes on the central site and increases linearly with the depth ℓ into the graph, $E_\ell = \sigma\ \ell$, with $\sigma > 0$.



(1.a) (**3 Points**) Define the micro canonical ensemble of states with a fixed energy $E = E_\ell$, and derive its entropy $S(E)$.

(1.b) (**3 Points**) From the entropy $S(E)$, calculate the temperature T of the ensemble, $1/T = dS/dE$. Distinguish the cases $z = 2$ and $z > 2$.

(1.c) (**4 Points**) Define the canonical ensemble at a given temperature T and calculate its free energy $F = E - TS$. For arbitrary T, minimize the free energy by optimizing the depth ℓ in the Bethe lattice. Which regimes do you find for $z = 2$ and $z > 2$ respectively?

Problem 2 Ideal classical gas – (solution: Tutorials)

Consider an ideal classical gas in $d = 3$ dimensions, described by the classical Hamiltonian

$$\hat{\mathcal{H}}^{(N)} = \sum_{n=1}^{N} \frac{\boldsymbol{p}_n^2}{2m} \tag{1}$$

where N denotes the number of particles of mass m.

(2.a) (**3 Points**) Calculate the partition function $Z_{\mathrm{C}}(N) = \mathrm{tr}\ e^{-\beta \hat{\mathcal{H}}^{(N)}}$ in the canonical ensemble.

(2.b) (**3 Points**) Calculate the partition function $Z_{\mathrm{GC}}(z) = \mathrm{tr}\ e^{-\beta\left(\sum_N \hat{\mathcal{H}}^{(N)} - \mu \hat{N}\right)}$ in the grand canonical ensemble, where $z = e^{\beta\mu}$ denotes the fugacity.

(2.c) (**4 Points**) Show for the grand canonical ensemble that the probability for finding a system with N particles is given by a Poisson distribution,

$$p(N) = e^{-\overline{N}} \frac{1}{N!} \overline{N}^N, \tag{2}$$

with average particle number $\overline{N}$. *Hint:* Start from the definition of the grand canonical partition function and use it as a generating functional to derive an expression for $p(N)$.

(2.d) (**2 Point**) Discuss how strongly the total particle number fluctuates in the the thermodynamic limit, $V \to \infty$ where $V = L^3$ is the system's volume.

Problem 3 Curie's law – **(solution: Central Exercise)**

Consider a system of N quantized spins in a magnetic field $\boldsymbol{B} = B\boldsymbol{e}_z$ at temperature T. The spins can be described by the Hamiltonian

$$\hat{\mathcal{H}} = -\mu B \sum_{i=1}^{N} m_i, \tag{3}$$

where μ is the magnetic moment, and $m_i = -s, -s+1, ..., s-1, s$ the quantized magnetization of the spins of length s.

The Gibbs free energy of this system is defined by

$$G(T, B, N) = -k_B T \log Z. \tag{4}$$

(3.a) (**3 Points**) Calculate the canonical partition function Z using a geometric series.

(3.b) (**2 Points**) Calculate the Gibb's free energy $G(T, 0, N)$ at $B = 0$.

(3.c) (**3 Points**) By Taylor expanding $G(T, B, N)$, one can show that

$$G(T, B, N) = G(T, 0, N) - \frac{Ns(s+1)\mu^2 B^2}{6k_B T} + \mathcal{O}(B^4). \tag{5}$$

Use this result to calculate the magnetization $M = \langle \mu \sum_{i=1}^{N} m_i \rangle$ and the magnetic susceptibility $\chi(T) = \frac{dM}{dB}|_{B=0}$. Show that $\chi \propto 1/T$ as expected from Curie's law.