TMP-TC2: Cosmology

Solutions to Problem Set 9 18 & 20 June 2024

1. Rotation curves of galaxies

We start from

$$v^2 = \frac{GM(r)}{r} \ , \tag{1}$$

where for a spherically symmetric distribution

$$M(r) = 4\pi \int_0^r dr' r'^2 \rho(r') \ , \tag{2}$$

with $\rho(r)$ the density, such that the object is only affected by the gravitational pull of the mass density inside its orbit.

1. We consider first the case where for $r < r_{core}$ the density is constant, i.e.

   $$\rho(r) \propto \rho_0 \Theta(r - r_{core}) \ ,$$

   with ρ_0 constant and $\Theta(x)$ is the Heaviside step function. Plugging the above into eqs. (**??**) and (**??**) we find that the circular velocity is

   $$v \propto \begin{cases} r \ , & r < r_{core} \\ r^{-1/2} \ , & r > r_{core} \end{cases}$$

   This corresponds roughly to the expectation curve from visible mass in the plot.
2. Consider now the possibility that the density obeys inverse power law, i.e.

   $$\rho \propto r^{-\alpha} \ , \ \alpha > 0 \ .$$

   In this case, eq. (**??**) gives us

   $$M(r) \propto r^{3-\alpha} \ , \ \alpha < 3 \ .$$

   Therefore, the circular velocity becomes

   $$v \propto r^{1-\alpha/2} \ .$$

3. For $r \geq r_{core}$, we want the velocity to be independent of r, which immediately gives $\alpha = 2$, whereas we would have expected $\alpha = 3$ for $r > r_{core}$ from what the mass density calculated from the observed luminosity of matter in the galaxy.

2. Weakly Interacting Massive Particles (WIMPs)

1. The annihilation of X particles switches off when their lifetime becomes of the order of the Hubble time,

$$\frac{1}{n_X \langle \sigma_{ann} v \rangle} = \frac{1}{H(T_f)},$$

where the annihilation cross-section in the non-relativistic regime can be approximated as

$$\sigma_{ann} = \frac{\sigma_0}{v}$$

The number densities in equilibrium at temperature $T < M_X$ are

$$n_X = n_{\bar{X}} = g_X \left(\frac{M_X T}{2\pi} \right)^{3/2} e^{-M_X/T}$$

Making use of these equations, we obtain the expression for the freeze-out temperature T_f,

$$\frac{1}{g_X \sigma_0} \left(\frac{2\pi}{M_X T_f} \right)^{3/2} e^{M_X/T_f} = \frac{M_P^*}{T_f^2}$$

where we assume that freeze-out occurs at radiation domination and used $M_P^* = \frac{M_P}{1.66 g_*^{1/2}}$. In the non-relativistic limit, one can infer with logarithmic accuracy

$$T_f = \frac{M_X}{\log\left(\frac{g_X M_X M_P^* \sigma_0}{(2\pi)^{3/2}} \right)}.$$

This temperature is smaller than M_X by a factor of

$$\left(\log \frac{g_X M_X M_P^* \sigma_0}{(2\pi)^{3/2}} \right)^{-1}.$$

2. The number density of X particles at freeze-out $n_X(t_f)$ is obtained by making use of Eq.(1),

$$n_X(t_f) = \frac{T_f^2}{M_P^* \sigma_0}$$

After freeze-out, n_X changes solely due to the cosmological expansion, so its present value is

$$n_X(t_0) = \left(\frac{a(t_f)}{a(t_0)} \right)^3 n_X(t_f).$$

We now use entropy conservation in comoving volume and write the above as

$$n_X(t_0) = \left(\frac{s_0}{s(t_f)}\right) n_X(t_f)$$

where $s(t_f)$ and s_0 are entropy densities at freeze-out and today. The present value of the entropy density is

$$s_0 = \frac{2\pi^2}{45}\left(2T_\gamma^3 + 6 \cdot \frac{7}{8}T_\nu^3\right) = 2.8 \cdot 10^3 \text{ cm}^{-3}.$$

Hence, the present number density of X particles is given by

$$n_X(t_0) = \frac{s_0 T_f^2}{s(t_f) M_P^* \sigma_0} = 3.8\frac{s_0}{T_f \sigma_0 M_P \sqrt{g_*(t_f)}},$$

where we used

$$s(t_f) = g_*(t_f) \cdot \frac{2\pi^2}{45} T_f^3.$$

3. For the relative mass density of X particles we have

$$\Omega_X = 2\frac{M_X n_X(t_0)}{\rho_c} = 7.6\frac{s_0 \log\left(\frac{g_x M_P^* M_X \sigma_0}{(2\pi)^{3/2}}\right)}{\rho_c \sigma_0 M_P \sqrt{g_*(t_f)}}$$

With known values of s_0 and ρ_c, we find numerically

$$\Omega_X = 3 \cdot 10^{-10}\left(\frac{GeV^{-2}}{\sigma_0}\right)\frac{1}{\sqrt{g_*(t_f)}}\log\left(\frac{g_X M_P^* M_X \sigma_0}{(2\pi)^{3/2}}\right)\frac{1}{2h^2}$$

Clearly, the strongest dependence here is on the parameter σ_0. The dependence on the mass M_X is logarithmic only, while the effective number of degrees of freedom g_* does not dramatically change during the cosmological evolution at freeze-out epoch.

To estimate the annihilation cross section we set $\sigma_0 \sim M_X^{-2}$ in the argument of logarithm in Eq.(14), on dimensional grounds. Setting $M_X = 100GeV$ and $g_* = 100$ for the estimate, we obtain the numerical value of the logarithmic factor,

$$\log\frac{g_X M_P^* M_X \sigma_0}{(2\pi)^{3/2}} \sim \log\frac{g_X M_P^*}{(2\pi)^{3/2} M_X} \sim 30$$

Since logarithm is a slowly varying function, this estimate is valid for wide range of masses M_X and cross sections σ_0. The uncertainty in the parameter $\sqrt{g_*(t_f)}$ is also moderate : at $T \gtrsim 100GeV$ and $T \sim 100MeV$ we have, respectively, $\sqrt{g_*(t_f)} \sim 10$ and $\sqrt{g_*(t_f)} \sim 3$. Thus, formula (14) gives the following estimate for the annihilation cross section

$$\sigma_0 \sim \frac{3 \cdot 10^{-10} \cdot 30GeV^{-2}}{(3-10) \cdot 0.25} = (0.36 - 1.2) \cdot 10^{-8}\text{GeV}^{-2} = (1.2 - 4) \cdot 10^{-36} \text{ cm}^2$$

Note that this value is comparable to weak interaction cross sections at energies of order $100GeV$, namely $\sigma_W \sim \alpha_W^2/M_W^2 \sim 10^{-7}G_eV^{-2}$. The result gives a cosmological lower bound on the annihilation cross section of hypothetical heavy stable particles that may be predicted by extensions of the Standard Model. If the cross section is below the value above, the mass density of these particles is unacceptably high. The main assumption behind this bound is that X particles were in equilibrium at some early epoch.

3. Primordial Black Holes as Dark Matter Candidates

1. First of all, they are mostly dark, since light trapped beyond their horizon could never reach us, and thus we could not detect them by luminosity observations.

   Second of all, they are matter. They have a mass which would create a gravitational pull, explaining the miss match with the rotational curves, and their density evolves as a^{-3}.

   Furthermore, the fact that they should mostly exist in galaxies and could be the seed to Supermassive Black Holes in the centre of galaxies is consistent with the observational evidence for the distribution of DM in the Universe. Moreover, they are nearly collisionless and have non-relativistic speeds, which is consistent with the fact that DM should not interact much with ordinary matter.

2. Two mass ranges are still open, $10^{-16}M_\odot \lesssim M \lesssim 10^{-10}M_\odot$ and $10^{14}M_\odot \lesssim M \lesssim 10^{18}M_\odot$.

3. First, we need check how Ω_{PBH} evolves with respect to the scale factor $a(t)$ from the time of the black holes formation until today. For a rough estimate, we will assume that the Universe was radiation dominated from the time of PBH formation until matter-radiation equality, and matter-dominated from then until now.

   Starting from the equation of state $\rho = wp$, where $w = \frac{1}{3}$ for radiation and $w = 0$ for matter. we can solve for the density in terms of the scale factor :

$$\dot{\rho} = -3H(\rho + p) = -3(1+w)H\rho$$

$$\implies \rho \propto a^{-3(1+w)}$$

   From which we have that

$$\Omega_{PBH} \propto \begin{cases} a & , w = \frac{1}{3} \\ 0 & , w = 0 \end{cases}$$

Which means that

$$\frac{\Omega_{PBH}(t_{eq})}{\Omega_{PBH}(t_F)} = \frac{a(t_{eq})}{a(t_F)} \qquad \text{and} \qquad \frac{\Omega_{PBH}(t_0)}{\Omega_{PBH}(t_{eq})} = 1$$

Recalling that $a(t) \propto t^{1/2}$ during radiation-domination and that $\Omega_{PBH}(t0) = \Omega_{PBH} \sim 0.25$, we calculate

$$\Omega_{PBH}(t_F) \sim \frac{\Omega_{DM}(t_0)}{\sqrt{\frac{t_{eq}}{1s)}}} \sim 10^{-7}$$

So notice we don't need to produce a lot of PBHs in the Early Universe for them to constitute all of DM.

4. We can turn the expression of the lifetime around to give us the mass based on the total evaporation time

$$M = \left(\frac{t_H M_P^4}{5120\pi\hbar}\right)^{\frac{1}{3}} . \tag{3}$$

So the mass of a PBH evaporating today is approximately 10^{14}g. This gives us the evaporation bound in the figure (in red), such that no PBH of mass smaller than 10^{14}g could explain any fraction of dark matter today. That is,

$$M \gtrsim 10^{14}\text{g} .$$

5. Extending the lifetime in such a way, we have that

$$M = \left(\frac{t_H M_P^6}{5120\pi\hbar}\right)^{\frac{1}{5}} , \tag{4}$$

such that the new evaporation bound becomes

$$M \gtrsim 5\text{x}10^6\text{g} ,$$

which of course extends the window of opportunity for PBHs as all of DM.