TMP-TC2: Cosmology

Solutions to Problem Set 8

11 & 13 June 2024

1. Big Bang Nucleosynthesis

Part A : Neutron-to-Proton Ratio

1. Protons and neutrons convert into each other through weak interactions, β-decay, inverse β-decay and neutron decay :

$$\begin{aligned} n + \nu_e &\leftrightarrow p^+ + e^- \\ n + e^+ &\leftrightarrow p^+ + \bar{\nu}_e \\ n &\leftrightarrow p^+ + e^- + \bar{\nu}_e \end{aligned}$$

2. Using the hint and the expression for the photon density in terms of the temperature, we have

$$n_{e^-} - n_{e^+} \sim n_B = \eta n_\gamma = \eta \frac{2}{\pi^2}\zeta(3)T^3$$

Furthermore, we can use the result from PS 5, for the ultra-relativistic limit :

$$n_{e^-} - n_{e^+} = \frac{2T^3}{6\pi^2}\left(\pi^2\left(\frac{\mu_e}{T}\right) + \left(\frac{\mu_e}{T}\right)^3\right)$$

Putting these two together, we see that since $\eta \sim 10^{-9}$ is very small, then we have

$$\frac{\mu_e}{T} \ll 1$$

3. Using again the result from PS 5 for the non-relativistic limit :

$$n_{e^-} - n_{e^+} = 2\left(\frac{mT}{2\pi}\right)^{3/2} e^{-\frac{m}{T}}\left(e^{\frac{\mu_e}{T}} - e^{-\frac{\mu_e}{T}}\right)$$

And so for $\mu_e \sim T$,

$$\eta\frac{2}{\pi^2}\zeta(3)T^3 \sim 2\left(\frac{mT}{2\pi}\right)^{3/2} e^{-\frac{m}{T}} \sinh 1$$

Solving this for T, we have $T \sim 20$keV. And so the $\frac{\mu_e}{T}$ is of order one for $T \sim 10 - 30$keV.

Thus, we can safely say that for $T \gg 20$keV, we are in the unltra-relativistic limit.

4. Looking at the processes we considered in Question 1, we see that when we neglect both the neutrino and the electron chemical potentials, we are left with the equilibrium condition $\mu_n \approx \mu_p$.

   Therefore, using Saha's equilibrium formula, we can take the ratio between the neutron and proton number densities to immediately give :

$$\left(\frac{n_n}{n_p}\right)_{\text{eq}} = \left(\frac{m_n}{m_p}\right)^{3/2} e^{-\frac{Q}{T}}, \tag{1}$$

   with $Q = m_n - m_p$.

   This result tells us that, for $T \gg Q$, we have as many neutrons as protons.

Part B : Neutron Freeze-out and Decay

1. At $T \sim$MeV, we can ignore the ratio between the proton and neutron mass, since $T \gg m_p$, and so the leading contribution comes from the exponential.

   The neutron abundance then becomes

$$X_n^{\text{freeze-out}} = X_n(T \sim 0.8\text{MeV}) = \frac{n_n}{n_n + n_p} \sim \frac{e^{-\frac{Q}{T}}}{1 + e^{-\frac{Q}{T}}} \sim 0.17 \sim \frac{1}{6}. \tag{2}$$

   Note : If we had calculated this exactly, by numerically solving the Boltzman equation, we would have found $X_n^{\text{freeze-out}} \sim 0.15$. So our very rough estimate gets us quite close!

2. For $T \sim 0.2$MeV, we have to start taking into account the decay of the neutrons with lifetime $\tau_n \approx 886.7s$, which just appears as an extra exponential decay term in the abundance :

$$X_n(t) = X_n^{\text{freeze-out}} e^{-\frac{t}{\tau_n}} \sim \frac{1}{6} e^{-\frac{t}{\tau_n}}.$$

Part C : Deuterium Bottleneck and Helium Production

1. To remove the dependance on the chemical potentials, we'll consider

$$\left(\frac{n_D}{n_p n_n}\right)_{\text{eq}} \approx \frac{g_D}{g_p g_n}\left(\frac{m_D}{m_p m_n}\frac{2\pi}{T}\right)^{\frac{3}{2}} \exp\left(\frac{-m_D+m_n+m_p}{T}\right)$$

Since $g_D = 3$ and $g_p = g_n = 2$ and we can estimate $m_n \sim m_p \sim \frac{1}{2}m_D$, we are left with

$$\left(\frac{n_D}{n_p}\right)_{\text{eq}} \approx \frac{3}{4}n_n^{\text{eq}}\left(\frac{4\pi}{m_p T}\right)^{\frac{3}{2}} e^{\frac{B_D}{T}}$$

where $B_D = m_n + m_p - m_D$ is the binding energy of Deuterium, as we wished to proved.

To estimate the order of magnitude of this ratio, take

$$n_n \sim n_b = \eta\frac{2}{\pi^2}\zeta(3)T^3 \sim 2.4\,10^{-10}\text{MeV}^3$$

And so for $T \sim 1\text{MeV} \gg B_D$, we can estimate

$$\left(\frac{n_D}{n_p}\right)_{\text{eq}} \sim 10^{-\frac{7}{2}}\eta.$$

But since $\eta = 10^{-9}$, we can see that this ratio is suppressed until the exponential term is large enough to compensate when $T \ll B_D$.

So we see that in Parts A and B, we did not have to worry about the production of the deuterium because the temperature was still to high for it to occur.

2. The synthesis ^{4}He starts when there are similar rates of protons and deuterium-nuclei present, that is, when the above ratio is of order 1 :

$$\left(\frac{n_D}{n_p}\right)_{\text{eq}} \sim \frac{3}{4}n_n^{\text{eq}}\left(\frac{4\pi}{m_p T}\right)^{\frac{3}{2}} e^{\frac{B_D}{T}} \sim 1 \longrightarrow T \sim 0.06\text{MeV}$$

3. Using the expression from Question 4 of Problem Set 5,

$$T \approx 1.56\; g_*^{-\frac{1}{4}}\sqrt{\frac{1\text{s}}{t}}\;\text{MeV}\;, \qquad (3)$$

we have that, with $g_* = 3.38$, $t_{\text{nuc}} \approx 340$s.

4. Let's compare the binding energies of Deuterium and Helium :

$$B_D \approx 2.2\text{MeV},$$
$$B_{^4\text{He}} \approx -1821\text{MeV}.$$

So the exponential term $e^{B/T}$ really favours the production of Helium over Deuterium!

This is referred to as the Deuterium Bottleneck. Because the binding energy of Deuterium is a lot higher, it takes much longer to produce enough deuterium for the nucleosynthesis of heavier elements to start. However, once there is enough Deuterium, the binding energy of Helium is so low that it's production is much quicker to start.

5. We are now almost ready to estimate the ratio of final abundances of helium and hydrogen, as we set out to do!

   First, we must estimate the neutron abundance at the time of nucleosynthesis

$$X_n(t_{\text{nuc}}) \sim \frac{1}{6} e^{-\frac{340}{886.7}} \sim 0.11 \sim \frac{1}{9}$$

   Since there are two neutrons in a Helium-4 nucleus, we can take $n_{He} \approx \frac{1}{2} n_n(t_{\text{nuc}})$

$$\longrightarrow \frac{n_{\text{He}}}{n_H} = \frac{n_{He}}{n_p} \sim \frac{\frac{1}{2} n_n(t_{\text{nuc}})}{n_p} = \frac{1}{2} \frac{X_n(t_{\text{nuc}})}{1 - X_n(t_{\text{nuc}})} \sim \frac{1}{16}$$

   Often, this is also written in terms of the mass fraction of Helium as

$$\frac{4 n_{He}}{n_H} \sim \frac{1}{4}$$

Part D : ^{4}He-Abundance

1. The weighted abundance of ^4He can be approximated as

$$X_{^4He} = \frac{2 \left. \frac{n_n}{n_p} \right|_{NS}}{1 + \left. \frac{n_n}{n_p} \right|_{NS}} \simeq 2 X_n \simeq 2 \exp\left\{ \left(-\frac{Q}{T^*} \right) \right\} , \tag{4}$$

   where $Q \equiv m_n - m_p$.

   We want to see how the abundance is affected when the number of the relativistic degrees of freedom changes, i.e. when the decoupling temperature is modified. From the above equation, we see that

$$\delta X_{^4He} = 2 \frac{Q}{T^*} \exp\left\{ \left(-\frac{Q}{T^*} \right) \right\} \frac{\delta T^*}{T^*} \tag{5}$$

   To find the change in the temperature, we use the fact that the reactions under consideration decouple at a temperature very close to the neutrino decoupling temperature, so

$$G_F^2 T^{*5} \sim \frac{T^{*2}\sqrt{g^*}}{\tilde{M}} \ . \tag{6}$$

Therefore,

$$\frac{\delta T^*}{T^*} = \frac{1}{6}\frac{\delta g^*}{g^*} \ . \tag{7}$$

Combining equations (4), (5) and (7) we get

$$\frac{\delta X_{^4He}}{X_{^4He}} = \frac{1}{6}\frac{Q}{T^*}\frac{\delta g^*}{g^*} \simeq \frac{1}{6}\frac{1.3}{1}\frac{\delta g^*}{g^*} \simeq 0.22\frac{\delta g^*}{g^*} \ . \tag{8}$$

2. If we consider an additional massless neutrino, we find that

$$\frac{\delta X_{^4He}}{X_{^4He}} \simeq 0.22\frac{\frac{7}{8}(2)}{2+\frac{7}{8}(4+6)} \simeq 3.6\% \ . \tag{9}$$

   A more detailed calculation combined with the uncertainty for the experimental measurements of $X_{^4He}$ gives an upper limit to the number of massless neutrinos

$$\#\nu < 4 \ .$$

3. In complete analogy with the above, we can calculate how $X_{^4He}$ varies when we change Q

$$\frac{\delta X_{^4He}}{X_{^4He}} \simeq -\frac{Q}{T^*}\frac{\delta Q}{Q} \simeq -1.3\frac{\delta Q}{Q} \simeq 13\% \ .$$

   Changing the neutron lifetime affects the ratio n_n/n_p, and so the neutron abundance X_n during nucleosynthesis. Using the expression from Part B,

$$\frac{\delta X_n}{X_n} = \frac{t_{NS}\delta\tau_n}{\tau_n^2}.$$

   Hence, since $X_{^4He} = 2X_n$, we have

$$\frac{\delta X_{^4He}}{X_{^4He}} \simeq \frac{20\text{mins}}{886.7\text{s}}\frac{\delta\tau_n}{\tau_n} \simeq 13\% \, .$$