TMP-TC2: Cosmology

Solutions to Problem Set 3

7 & 9 May 2024

1. Universe evolutions

— *Universe composed of radiation and a cosmological constant $\lambda > 0$*

The Friedmann equations in this case read

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{\lambda}{3} = \frac{8\pi G}{3}\rho$$

$$2\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 - \lambda = -\frac{8\pi G}{3}\rho.$$

We sum the above to obtain a differential equation for $R(t)$

$$\ddot{R}R + \dot{R}^2 = \frac{2\lambda}{3}R^2 \; .$$

It is easy to see that we can write it as

$$\frac{1}{2}\frac{d^2}{dt^2}\left(R^2\right) = \frac{2\lambda}{3}R^2 \; ,$$

and by introducing $x = R^2$, the equation becomes

$$\ddot{x} = \frac{4\lambda}{3}x \; .$$

The general solution is

$$x(t) = c^2 \sinh\left(\sqrt{\frac{4\lambda}{3}}\, t + \varphi_0\right) \; ,$$

where c and φ_0 depend on the initial conditions. Then, the scale factor is

$$R(t) = c \sinh^{\frac{1}{2}}\left(2\sqrt{\frac{\lambda}{3}}\, t + \varphi_0\right) \; .$$

Requiring that $R(0) = 0$ results into $\varphi_0 = 0$, so the final result is

$$R(t) = c \sinh^{\frac{1}{2}}\left(2\sqrt{\frac{\lambda}{3}}\, t\right) \; .$$

— *Universe composed of matter and a cosmological constant $\lambda > 0$*

In this case we can immediately work with the second Friedmann equation that now reads

$$\ddot{R}R + \frac{1}{2}\dot{R}^2 = \frac{\lambda}{2}R^2 \ . \tag{1}$$

To solve it, we make the following ansatz

$$R(t) = c \sinh^{\alpha}(\beta t) \ , \tag{2}$$

with α and β constants to be determined by requiring that the above is indeed a solution of (1). A simple computation gives us

$$\alpha\beta^2 \left(4 - 3\alpha - 3\alpha\cosh(2\beta t)\right) + 2\lambda \sinh^2(\beta t) = 0 \ . \tag{3}$$

Using the well-known

$$\cosh 2x = 2\cosh^2 x - 1 \ , \quad \text{and} \quad \cosh^2 x - \sinh^2 x = 1 \ , \tag{4}$$

we find that

$$\alpha = \frac{2}{3} \ , \quad \beta = \frac{\sqrt{3\lambda}}{2} \ . \tag{5}$$

Therefore, the final result is

$$R(t) = c \sinh^{\frac{2}{3}}\left(\frac{3}{2}\sqrt{\frac{\lambda}{3}}\, t\right) \ .$$

To find the age of the universe as a function of H_0 and Ω_m we proceed as follows. First, using the last relation, we compute the Hubble parameter for $t = t_0$

$$H_0 = \frac{\dot{R}(t_0)}{R(t_0)} = \sqrt{\frac{\lambda}{3}} \coth\left(\frac{3}{2}\sqrt{\frac{\lambda}{3}}\, t_0\right) \ .$$

Inverting this expression and using the hint, we get

$$t_0 = \frac{2}{3}\sqrt{\frac{3}{\lambda}} \operatorname{arccoth}\left(\sqrt{\frac{3H_0^2}{\lambda}}\right) = \frac{1}{3}\sqrt{\frac{3}{\lambda}} \ln \frac{\sqrt{\frac{3H_0^2}{\lambda}} + 1}{\sqrt{\frac{3H_0^2}{\lambda}} - 1} = \frac{1}{3}\sqrt{\frac{3}{\lambda}} \ln \frac{1 + \sqrt{\frac{\lambda}{3H_0^2}}}{1 - \sqrt{\frac{\lambda}{3H_0^2}}} \ .$$

Then we have to trade λ for Ω_m. To this end, we use the first Friedmann equation for $t = t_0$

$$H_0^2 = \frac{\lambda}{3} + \frac{8\pi G}{3}\rho(t_0) \ .$$

Dividing both sides by H_0^2 we find

$$\frac{\lambda}{3H_0^2} = 1 - \Omega_m \ .$$

Therefore,

$$\begin{aligned} t_0 &= \frac{1}{3H_0}\frac{1}{\sqrt{1-\Omega_m}} \ln \frac{1 + \sqrt{1-\Omega_m}}{1 - \sqrt{1-\Omega_m}} \\ &= \frac{1}{3H_0}\frac{1}{\sqrt{1-\Omega_m}} \ln \frac{(1 + \sqrt{1-\Omega_m})^2}{\Omega_m} \\ &= \frac{2}{3H_0}\frac{1}{\sqrt{1-\Omega_m}} \ln \frac{1 + \sqrt{1-\Omega_m}}{\sqrt{\Omega_m}} \ . \end{aligned}$$

2. The fate of the universe

1. We start from the first Friedmann equation written using abundances, i.e.

$$\Omega_{\text{mat}} + \Omega_k + \Omega_\lambda = 1.$$

It is clear that the line $\Omega_\lambda = 1 - \Omega_{\text{mat}}$ corresponds to a flat universe. Above this curve, we have $k = 1$ and below it $k = -1$.
We now consider the second Friedmann equation

$$\frac{\ddot{R}}{R} + \frac{4\pi G}{3}\rho - \frac{\lambda}{3} = 0,$$

that in terms of the abundances becomes

$$\Omega_\lambda = \frac{\Omega_{\text{mat}}}{2} + \frac{\ddot{R}}{RH^2}.$$

The curve $\Omega_\lambda = \Omega_{\text{mat}}/2$ describes a universe with zero acceleration. So, we have an accelerating universe above the curve and a deccelerating one below.

2. In this case, the Friedmann equations simplify considerably and read

$$\begin{cases} \dfrac{\ddot{R}}{R} = \dfrac{\lambda}{3}, \\ \dot{R}^2 + k = \dfrac{\lambda}{3} R^2. \end{cases}$$

By solving the first equation, we obtain constraints on the curvature k.
$\underline{\lambda > 0 :}$ In this case, the solution is given by

$$R(t) = A \exp\left(\sqrt{\frac{\lambda}{3}}\, t\right) + B \exp\left(-\sqrt{\frac{\lambda}{3}}\, t\right).$$

By inserting this expression in the second equation, we obtain

$$k = \frac{4AB}{3}\,\lambda.$$

This constraint tell us, that for $k = 1$, A and B have to be both positive or negative. Then, an initial singularity is impossible. On the other hand, if $k = -1$, A and B have to be of opposite sign, and an initial singularity is possible for $A = -B = \sqrt{3/(4\lambda)}$. The solution is

$$R(t) = \sqrt{\frac{3}{\lambda}} \sinh\left(\sqrt{\frac{\lambda}{3}}\, t\right).$$

$\underline{\lambda < 0 :}$ In this case, the solution is given by

$$R(t) = A \cos\left(\sqrt{\frac{|\lambda|}{3}}\, t\right) + B \sin\left(\sqrt{\frac{|\lambda|}{3}}\, t\right).$$

The second equation gives

$$k = \frac{\lambda}{3}\left(A^2 + B^2\right).$$

Since $\lambda < 0$, k has to be -1. Then, for $A = 0$, $B = \sqrt{3/|\lambda|}$, we have an initial singularity. We thus get

$$R(t) = \sqrt{\frac{3}{|\lambda|}}\sin\left(\sqrt{\frac{|\lambda|}{3}}\, t\right).$$

3. If we neglect the cosmological constant, the first Friedmann equation is

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho.$$

Since the energy of the universe is conserved, ρR^3 is constant and we use it to express ρ as a function of R. It is convenient to introduce a new variable $r^2 = \left(R/R_0\right)^2$ to recase the Friedmann equation as

$$\dot{r}^2 = \frac{8\pi G}{3}\frac{\rho_0}{r} - \frac{k}{R_0^2}$$

Using the definitions of the abundances, we trade Ω_k for Ω_λ and Ω_{mat} to find from the above

$$\dot{r}^2 = H_0^2\left[\Omega_m^0\left(\frac{1}{r} - 1\right) + 1\right],$$

or in other words

$$E_{kin} + U(r) = E_{tot},$$

with

$$E_{kin} = \dot{r}^2\ , \quad U(r) = -\frac{H_0^2\Omega_m^0}{r},$$

and

$$E_{tot} = H_0^2\left(1 - \Omega_m^0\right).$$

The potential $U(r)$ is monotonic, negative and $U(r) \to 0$, when $r \to \infty$. Then, we first deduce, that for all possible values of E_{tot}, there is an initial singularity. Secondly, depending on the sign of E_{tot}, we have infinite expansion or a collapse in the future. For $E_{tot} > 0$ i.e. $\Omega_m^0 < 1$, we have infinite expansion. Otherwise, the universe eventually collapses in the future.

4. We proceed exactly as before. We write the first Friedmann equation as

$$E_{kin} + U(r) = 0,$$

with

$$U(r) = -\frac{H_0^2}{2}\left[\Omega_m^0\left(\frac{1}{r} - 1\right) + \Omega_\lambda^0\left(r^2 - 1\right) + 1\right].$$

We will study this potential in the cases where $\Omega_\lambda < 0$ and $\Omega_\lambda > 0$ separately.
$\underline{\Omega_\lambda < 0:}$ First, note that

$$\lim_{r\to 0} U(r) = 0 \ , \quad \lim_{r\to\infty} U(r) = \infty.$$

Computing the derivative of the potential

$$U'(r) = -\frac{H_0^2}{2}\left[2\Omega_\lambda^0 r - \frac{\Omega_m^0}{r^2}\right] > 0,$$

we notice that it is a monotonic function. Then, there is an initial singularity and the universe grows until r_b (for which $U(r_b) = 0$), then it collapses.
$\underline{\Omega_\lambda > 0:}$ In this case,

$$\lim_{r\to 0} U(r) = -\infty \ , \quad \lim_{r\to\infty} U(r) = -\infty,$$

meaning that the potential has a maximum :

$$U'(r_{max}) = -\frac{H_0^2}{2}\left[2\Omega_\lambda^0 r_{max} - \frac{\Omega_m^0}{r_{max}^2}\right] = 0,$$

therefore

$$r_{max} = \left(\frac{\Omega_m^0}{2\Omega_\lambda^0}\right)^{1/3}.$$

Then

$$U_{max} = U(r_{max}) = -\frac{H_0^2}{2}\left[\frac{3}{2^{2/3}}(\Omega_m^0)^{2/3}(\Omega_\lambda^0)^{1/3} + (1 - \Omega_m^0 - \Omega_\lambda^0)\right].$$

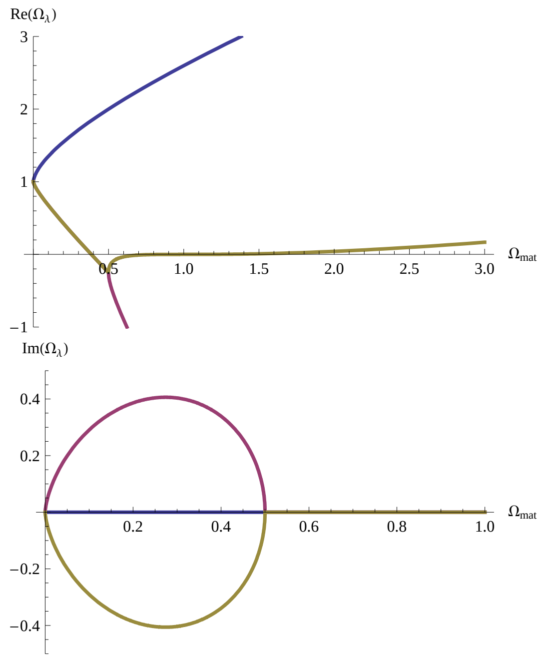
The solutions of the equation $U_{max} = 0$ are given by

$$\begin{cases} \Omega_{\lambda,1} = 1 - \Omega_m + \dfrac{3}{2}\left(\dfrac{\Omega_m^2}{D^{1/3}} + D^{1/3}\right), \\ \Omega_{\lambda,2} = 1 - \Omega_m - \dfrac{3}{4}\left(\dfrac{(1+i\sqrt{3})\Omega_m^2}{D^{1/3}} + (1-i\sqrt{3})D^{1/3}\right), \\ \Omega_{\lambda,3} = 1 - \Omega_m - \dfrac{3}{4}\left(\dfrac{(1-i\sqrt{3})\Omega_m^2}{D^{1/3}} + (1+i\sqrt{3})D^{1/3}\right) \end{cases}$$

with

$$D = \Omega_m^2 - \Omega_m^3 + \sqrt{\Omega_m^4 - 2\Omega_m^5}.$$

The real and imaginary parts of the three solutions are plotted in the following two graphs :



From these graphs we observe the following : $\Omega_{\lambda,1}$ (blue curve) is real and positive for all values of Ω_m, so this solution is physical. $\Omega_{\lambda,2}$ (red curve) is complex for $\Omega_m < 0.5$ and becomes real and negative for $\Omega_m > 0.5$, so it is an unphysical solution. Finally, $\Omega_{\lambda,3}$ (yellow curve) is complex for $\Omega_m < 0.5$, it is real and negative for $0.5 < \Omega_m < 1$, and it is real and positive for $\Omega_m > 1$, meaning that it is physical from $\Omega_m > 1$.

Above the first curve ($\Omega_{\lambda,1}$), we have $U_{max} > 0$ and $r_{max} < 1 = r_0$(today), then there is no initial singularity and the universe expands forever.

Between the two solutions, we have $U_{max} < 0$, then there is an initial singularity and the universe expands forever.

Finally, under the second curve ($\Omega_{\lambda,3}$), we have $U_{max} > 0$ and $r_{max} > 1 = r_0$(today), then there is an initial singularity and the universe will collapse in the future.

Remark : Experimentally, it has been measured :

$$\Omega_m = 0.24 \pm 0.04,$$

$$\Omega_\lambda = 0.76 \pm 0.06,$$

$$\Omega_{total} = 1.003 \pm 0.017.$$

The universe has an initial singularity, it will expand forever and is accelerating. It is not known, whether the universe is closed or open, $\Omega_k = -0.003 \pm 0.017$.

3. Recollapsing Universe

1. The first Friedmann equation can be rewritten as

$$\frac{\dot{R}^2}{R^2} = \frac{R_m}{R^3} - \frac{1}{R^2} \,, \tag{6}$$

with

$$R_m = \frac{8\pi G}{3}\rho R^3 \ .$$

For a recollapsing universe, there is a time when the expansion stops, i.e. $\dot{R} = 0$. Then

$$R_m = R \ . \tag{7}$$

2. From the Friedmann equation (6), we find

$$\dot{R} = \pm\sqrt{\frac{R_m}{R} - 1} \ , \tag{8}$$

translating into

$$dt = \pm\frac{dR}{\sqrt{\frac{R_m}{R} - 1}} \ . \tag{9}$$

Integrating this equation with positive sign leads to

$$t = -R\sqrt{\frac{R_m}{R} - 1} - R_m \arctan(\sqrt{\frac{R_m}{R} - 1}) + R_m\frac{\pi}{2} \tag{10}$$

where we used that $R(0) = 0$ to determine the integration constant. The universe reaches its maximal size for $R = R_m$, which corresponds to $t_m = \frac{\pi}{2}R_m$. After this point we must continue the solution by taking the negative sign in (10). Therefore, the total lifetime is

$$t_{\text{life}} = \pi R_m \tag{11}$$

3. The Friedmann equations in that case read

$$H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}$$
$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\rho + \frac{\Lambda}{3}$$

Combining the above we find

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} = \Lambda \ , \tag{12}$$

with solution

$$R(t) = R_m \sin^{\frac{2}{3}}\left(\frac{\sqrt{3|\Lambda|}}{2}t\right) \tag{13}$$

The maximal expansion is reached for $\sin(\ldots) = 1$, translating into $t_m = \frac{\pi}{\sqrt{3|\Lambda|}}$. The lifetime of the universe is

$$t_{\text{life}} = \frac{2\pi}{\sqrt{3|\Lambda|}} \tag{14}$$