TMP-TC2: Cosmology

Problem Set 3 7 & 9 May 2024

1. Universe evolutions

Consider a homogeneous and flat Universe described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = dt^2 - R(t)^2 \left(dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right) \ .$$

Find the scale factor $R(t)$ for the following cases :

— Universe composed of radiation and cosmological constant $\lambda > 0$.

— Universe composed of matter (non-relativistic) and cosmological constant $\lambda > 0$.

For the last case, verify that the age of the universe is given by :

$$t_0 = \frac{2}{3H_0}\frac{1}{\sqrt{1-\Omega_m}} \ln \frac{1+\sqrt{1-\Omega_m}}{\sqrt{\Omega_m}} \ ,$$

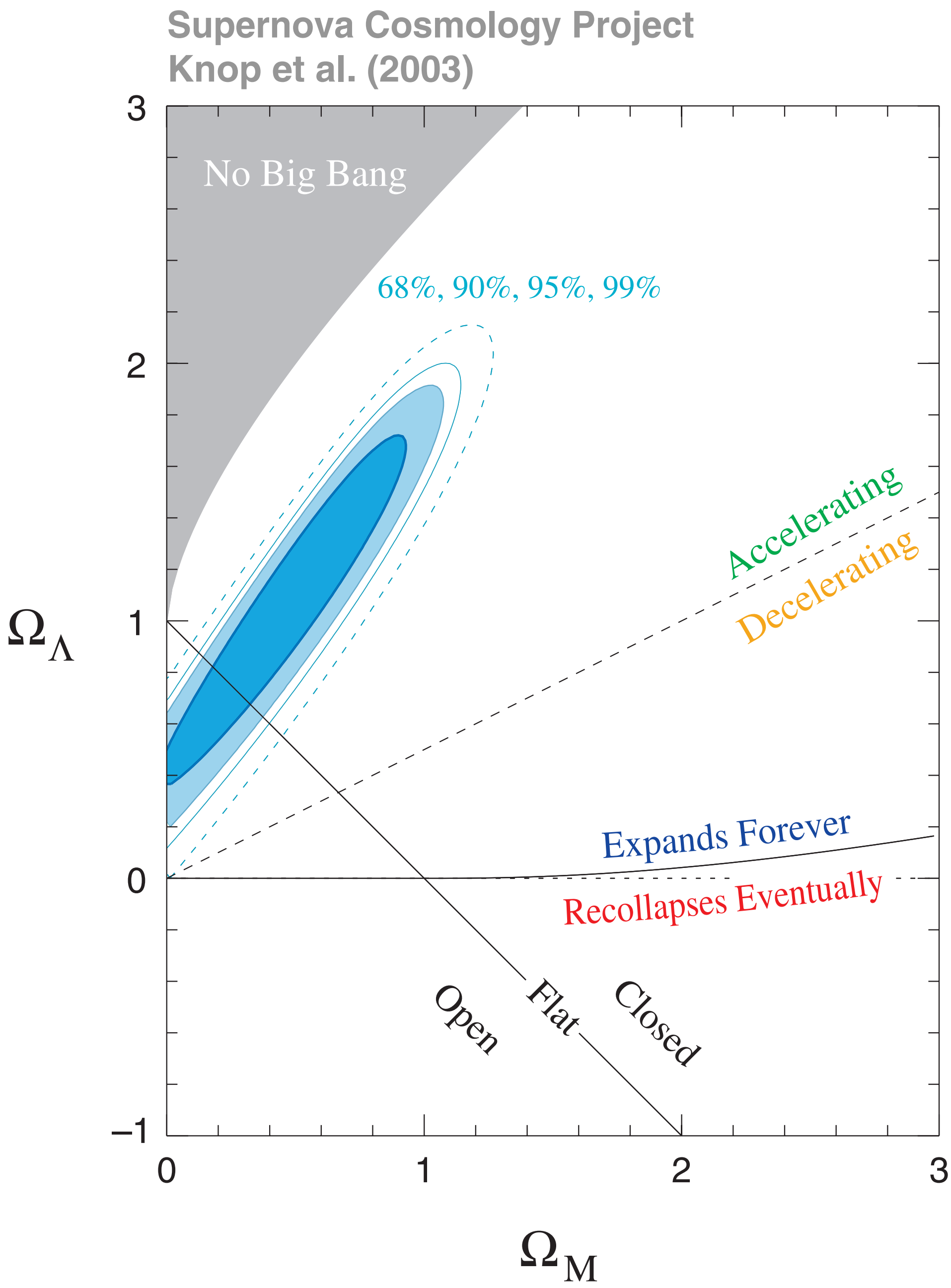
where $\Omega_m = \frac{\rho_m(t_0)}{\frac{3H_0^2}{8\pi G}}$.

Indication: : you may use $\text{arccoth} x = \frac{1}{2}\ln\frac{x+1}{x-1}$

2. The fate of the Universe

The purpose of this exercise is to study the different areas and limits in the graph $(\Omega_m, \Omega_\lambda)$ shown below. ($\Omega_\lambda = \frac{\lambda}{3H_0}$)

1. Identify areas where the Universe is open or closed and those where it is accelerating or decelerating.
2. In the case of a Universe dominated by the cosmological constant λ, ($\rho = 0$, $p \approx 0$), identify the areas where there was a singularity in the past (Big Bang).
   Indication: Consider the cases $\lambda > 0$ and $\lambda < 0$ separately.
3. In the case of a Universe dominated by matter, ($p \approx 0$, $\lambda = 0$), show that there is an initial singularity. For which values of Ω_m do we have a collapse in the future ?
   Indication: You may need $\rho R^3 = \text{const}$. Argue why this is constant.
4. For $\rho \neq 0$ and $\lambda \neq 0$, identify areas where there is no initial singularity as a function of today's parameters Ω_λ^0, Ω_m^0 and H_0, and those where the universe expands or collapses in the future.
   Indication: You may want to use Mathematica to get the precise results. Otherwise, discuss qualitatively the different possibilities.



3. Recollapsing Universe

The relative contributions of different terms in the r.h.s. of Friedmann equation (non-relativistic matter, radiation, cosmological constant) to the total energy density change as the Universe evolves. In particular, the terms with different signs can dominate at different stages of the evolution, leading to an expanding Universe at earlier times and a collapsing one at later times. As an example, consider a closed ($k = 1$) Universe filled with non-relativistic matter. The Friedmann equation reads as follows,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{R_m}{R^3} - \frac{1}{R^2}, \tag{1}$$

where R_m is some constant.

1. Show that R_m is the largest size of the recollapsing Universe (i.e. the size at

which $\dot{R} = 0$). Relate it to the total mass of the non-relativistic matter. Find R_m numerically when the Universe is filled with $1kg$ of matter.

2. Solve Eq.(1) and find the total lifetime of the Universe.
3. Suppose now that the Universe is flat ($k = 0$) and filled with non-relativistic matter and negative cosmological constant. Find $R(t)$ in this case. Find the total lifetime.