TMP-TC2: Cosmology

Solutions to Problem Set 11 2 & 4 July 2024

1 Flatness Problem

Using the given equation $\Omega - 1 = \frac{k}{R^2 H^2}$ we can find for an arbitrary time t :

$$|\Omega(t) - 1| = \frac{R_0^2 H_0^2}{R(t)^2 H(t)^2} |\Omega_0 - 1| \tag{1}$$

With $R(t) \propto t^n$ ($n = \frac{1}{2}$ for radiation domination and $n = \frac{2}{3}$ for matter domination) we get

$$|\Omega(t) - 1| = \left(\frac{t}{t_0}\right)^{2(1-n)} |\Omega_0 - 1| \tag{2}$$

Inserting the time of recombination $t_R \approx 3.7 \cdot 10^5$ years, we obtain

$$|\Omega(t_R) - 1| \approx \mathcal{O}(10^{-8} - 10^{-9}) \tag{3}$$

It seems that this number is very fine-tuned and surprisingly close to the value zero corresponding to a flat universe. But why ? This is the flatness problem.

2 Horizon Problem

Let us assume that at one point in the past, a signal was emitted. Then the proper distance between the observer and the source is given at time t_0 by

$$d(t_0) = R(t_0) \int_{t_e}^{t_0} \frac{1}{R(t)} \mathrm{d}t \tag{4}$$

If t_e is the time of the emission of the CMB and t_0 is the age of the universe today, then d describes the distance between us and the CMB.
The size of the causally connected region at t_e is

$$D(t_0) = R(t_0) \int_{0}^{t_e} \frac{1}{R(t)} \mathrm{d}t \tag{5}$$

Then the angle that contains one causally connected region in the sky is

$$\theta = 2 \arctan \left(\frac{1}{2} \frac{D(t_0)}{d(t_0)} \right) \tag{6}$$

For a matter dominated universe we have $R(t) \propto t^{\frac{2}{3}}$. Therefore, we obtain for D and d

$$D(t_0) = 3 t_0^{\frac{2}{3}} t_e^{\frac{1}{3}} \tag{7}$$

$$d(t_0) = 3 t_0^{\frac{2}{3}} (t_0^{\frac{1}{3}} - t_e^{\frac{1}{3}}) \tag{8}$$

With $1 + z = \frac{R(t_0)}{R(t_e)} = \left(\frac{t_0}{t_e} \right)^{\frac{2}{3}}$ we obtain for the angle

$$\theta = 2 \arctan \left(\frac{1}{2} \frac{1}{\sqrt{1+z} - 1} \right) \tag{9}$$

With $z \approx 1500$, we get the angle $\theta \approx 1.52°$.
The problem with this small angle is that in the CMB are many causally disconnected patches. However, the CMB is very isotropic. How can this be ? One solution to this is for example inflation. We will discuss this on the next sheet.

3. Equations of motion for a homogeneous scalar field in FLRW

We have a theory described by the following action

$$S = S_g + S[\phi] \ ,$$

where the gravitational part S_g is the usual Einstein-Hilbert action

$$S_g = \int d^4x \frac{1}{16\pi G} R \ ,$$

and the inflaton's part is

$$S[\phi] = \int d^4x \sqrt{-g} \ \mathcal{L}[\phi] = \int d^4x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] \ .$$

The variation of the action with respect to ϕ yields

$$\begin{aligned} \delta_\phi S[\phi] &= \int d^4x \sqrt{-g} \left(-g^{\mu\nu} \partial_\nu \phi \partial_\mu \delta\phi - V'(\phi) \delta\phi \right) \\ &= \int d^4x \left[\partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right) - \sqrt{-g} V'(\phi) \right] \delta\phi \ , \end{aligned}$$

so the equation of motion for the field is

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right] - V'(\phi) = 0 \ .$$

We now introduce to the above the explicit form of the (spatially flat) FLRW metric

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2) \quad \rightarrow \quad g_{\mu\nu} = \mathrm{diag}[-1, a(t)^2, a(t)^2, a(t)^2] \ ,$$

and use $g = -a(t)^6$, therefore

$$\frac{1}{a(t)^3}\partial_\mu\left[a(t)^3 g^{\mu\nu}\partial_\nu\phi\right] - V'(\phi) = 0 \ \rightarrow \ \frac{1}{a(t)^3}\partial_0\left[a(t)^3 g^{00}\partial_0\phi\right] + g^{ii}\partial_i\partial_i\phi - V'(\phi) = 0 \ .$$

If we take into account that the inflaton is homogeneous, the spatial derivatives can be neglected so the above gives us

$$\ddot{\phi} + 3H\phi + V'(\phi) = 0 \ ,$$

where $H = \dot{a}/a$ is the Hubble parameter.
The second equation comes from the variation of the action with respect to the metric. We have seen that the variation of the Einstein-Hilbert action S_g with respect to the metric yields the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - 1/2 g_{\mu\nu}R$. The variation of the action for the scalar field $S[\phi]$ with respect to the metric is

$$\delta_g S_\phi = \int d^4x \left(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\ \mathcal{L}[\phi] - \frac{1}{2}\sqrt{-g}\partial_\mu\phi\partial_\nu\phi\right)\delta g^{\mu\nu} \ .$$

Identifying the energy-momentum tensor with

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S[\phi]}{\delta g^{\mu\nu}} \ ,$$

we get

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\mathcal{L}[\phi] \ .$$

Putting everything together we get Einstein's equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \ .$$

The above for the 00 component in the FLRW space (look also at the Problem Set 1) is

$$G_{00} = 8\pi G T_{00} \quad \rightarrow \quad 3H^2 = 8\pi G\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) ,$$

where we neglected again the spatial variations of ϕ.

4. Scalar field in FLRW spacetime

1. We have seen that the energy momentum tensor $T_{\mu\nu}$ of a scalar field ϕ reads

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left[\frac{1}{2} \left(\partial_\kappa \phi \right)^2 - V(\phi) \right]$$

   where $V(\phi)$ is the potential.

   The energy density of the field corresponds to

$$\rho = T_{00} = \frac{\dot{\phi}^2}{2} + \frac{1}{2a^2} \left(\partial_i \phi \right)^2 + V(\phi)$$

   where dot denotes derivative with respect to time and a is the scale factor. The pressure p is related to the spatial components of the energy-momentum tensor as

$$p = \frac{1}{3} T_i^i = \frac{\dot{\phi}^2}{2} - \frac{1}{6a^2} \left(\partial_i \phi \right)^2 - V(\phi)$$

   Using the above, we see that the equation of state parameter is

$$w \equiv \frac{p}{\rho} = \frac{\frac{\dot{\phi}^2}{2} - \frac{1}{6a^2} \left(\partial_i \phi \right)^2 - V(\phi)}{\frac{\dot{\phi}^2}{2} + \frac{1}{2a^2} \left(\partial_i \phi \right)^2 + V(\phi)}$$

2. Accelerated expansion requires

$$p < -\frac{\rho}{3}$$

   For the scalar field, the above condition gives us

$$\dot{\phi}^2 < V(\phi)$$

   which means that the potential energy of the field must dominate over its kinetic energy.

3. If we assume that the field is homogeneous, i.e. $\phi \equiv \phi(t)$, the expressions for ρ and p we found before simplify significantly

$$\rho = \frac{\dot{\phi}^2}{2} + V(\phi) \quad \text{and} \quad p = \frac{\dot{\phi}^2}{2} - V(\phi)$$

Using the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0$$

we immediately find

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0$$

which is the Klein-Gordon equation of a (homogeneous) scalar field in a flat FLRW universe.