Ludwig-Maximilians-Universität München

TMP-TC2 : COSMOLOGY

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Guidelines :

- The exam consists of 7 problems. You must solve problems 1 - 4.
  You must choose only one of the remaining problems 5 - 7 to solve.
- You have to upload your solutions to sync+share by Friday 28 July 20.00 CEST, at the latest.
- Please write your name AND matriculation number on every sheet that you hand in.
- You can look up in your favorite resource the values of the numerical parameters that you need.
- Your answers should be comprehensible and readable.
- It goes without saying that we expect you to work independently.

GOOD LUCK !

Problem 1	40 P
Problem 2	20 P
Problem 3	20 P
Problem 4	10 P
Extra Problem	30 P

Total	120 P

Problem 1 (40 points)

1. Write down the Friedman equations for a flat universe with cosmological constant and radiation.
2. Calculate the number of effective degrees of freedom when the universe had temperature $2GeV$.
3. Explain where the following estimation for the decoupling temperature is coming from
$$H(T_d) \sim \Gamma(T_d) \ . \tag{1}$$
4. At what temperature of the universe, does the neutron decay become important ? You can do a short calculation.
5. What is the Deuterium bottleneck ?
6. How many neutrinos produced in the early universe pass through your hand per second ? You can do a short calculation.
7. Explain the Horizon problem.
8. What is a zero mode of a soliton ? Briefly discuss it with an example of your choice.
9. Do we know if there is a magnetic monopole problem ? Explain it. If there is a magnetic monopole problem, what are the solutions to this ? Name at least one.
10. Do you know a mechanism to produce primordial black holes ?

Problem 2 (20 points)

Consider the following action capturing the dynamics of a homogeneous scalar field ϕ
$$S[\phi, g] = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \ , \tag{2}$$
which evolves in an FLRW spacetime with line element
$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2\right] \ , \tag{3}$$
and $k \neq 0$.

1. Find the equations of motion for ϕ.
2. Find the Friedmann equations and determine the expressions for the pressure and energy density in terms of ϕ.
Hint : You are allowed to use the results that we found in the tutorials.

Problem 3 (20 points)

Consider a fermion ψ with mass $m = 10$ GeV that at temperatures $T \gg m$ is maintained in thermal equilibrium by the reaction $\overline{\psi}\psi \leftrightarrow \gamma\gamma$. It decouples at $T = 50$ MeV.

1. What is the number $g_\star$ of the effective relativistic degrees of freedom at decoupling ?
2. What is the cross-section of the reaction mentioned above at decoupling ? Assume that the fermion has zero chemical potential.

Problem 4 (10 points)

Recently, the NANOGrav experiment detected gravitational waves with low frequencies of the order of $\omega_0 = 1nHz$. Assuming that this signal was emitted when the temperature of the Universe was $T_{\text{em}} = 10^{16}GeV$, compute the frequency ω_{em} of the gravitational waves at the time of emission. Assume a radiation-dominated universe.

YOU MUST CHOOSE <u>ONLY ONE</u> OF THE FOLLOWING PROBLEMS

Problem 5 (30 points)

A standard prediction of Grand Unified Theories is the existence of magnetic monopoles with mass $m \simeq 10^{16}GeV$.

1. Suppose that the universe is dominated by radiation. At temperature $T_m = 10^{16}GeV$ assume the existence of one monopole per horizon $r \simeq M_{Pl}/T_m^2$. Calculate the current energy density of the monopole, assuming that it was not annihilated and also no more monopoles were created.

2. Suppose now that the radiation-dominated epoch ended at $T_i = 10^{10}GeV$ and the Universe underwent a period of exponential expansion with scale factor $a \propto e^{Ht}$, and $H = 10^{25}s^{-1}$. After this period, the universe

got reheated instanteneously to $T_r = 10^{10} GeV$, and continued to be dominated by radiation until $T_e = 10^{-10} GeV$, after which it became dominated by matter until today. How long should the period of exponential expansion last in order to have the energy density of the monopole smaller than the critical density ?

Problem 6 (30 points)

Consider a spacetime with the following line element

$$ds^2 = \left(\frac{\tau}{\tau_0}\right)^2 \left[-d\tau^2 + dx^2 + dy^2 + dz^2\right] \ , \tag{4}$$

with τ_0 a constant.

1. Write $g_{\mu\nu}$ and determine $g^{\mu\nu}$.
2. Compute the nonvanishing Christoffel symbols $\Gamma^{\lambda}{}_{\mu\nu}$.
3. Compute the nonvanishing components of the Riemann tensor $\mathcal{R}^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}{}_{\nu\sigma} + \Gamma^{\mu}{}_{\kappa\rho}\Gamma^{\kappa}{}_{\nu\sigma} - (\rho \leftrightarrow \sigma)$.
4. Compute the Ricci tensor $\mathcal{R}_{\mu\nu}$ and the scalar curvature $\mathcal{R}$.
5. Is this spacetime flat ? Justify your answer.
6. What is this Universe dominated by ? Justify your answer.
   Hint : You may bring the line element into the usual form

   $$ds^2 = -dt^2 + R^2(t)[dx^2 + dy^2 + dz^2] \ ,$$

   with $t = t(\tau)$.

Problem 7 (30 points)

Black Holes can form from the collapse of vacuum bubbles that spontaneously nucleate at the end of inflation. Outside the bubble, the Universe is described by the FLRW metric with scale factor $R(t)$. Note that we will use a dot $\dot{}$ to refer to derivatives with respect to proper time η defined as $d\eta = dt/R$ and the index i will be use to refer to quantities evaluated at time η_i corresponding to the end of inflation.

The mass of the bubble can be approximated as

$$M \approx \frac{4}{3}\pi(\rho_b - \rho)r^3 + 4\pi\sigma r^2 \left(\dot{r}^2 + 1\right)^{1/2} \ , \tag{5}$$

where $r = r(\eta)$ represents the radius of the bubble, $\rho = \rho(\eta)$ is the vacuum energy density outside the bubble

$$\rho(\eta) = \frac{8\pi G}{3} H(\eta)^2 \ , \tag{6}$$

σ corresponds to the wall tension of the bubble and $\rho_b(\eta)$ is the vacuum energy density inside the bubble.

1. What can we say about the vacuum energy density outside the bubble $\rho_i = \rho(\eta_i)$ with respect to $\rho_b(\eta_i)$ in order for bubbles to spontaneously nucleate ?
2. Argue that we can approximate the speed at which the wall of the bubble moves initially as

   $$\dot{r}_i \sim \frac{1}{8\pi G\sigma} H_i^2 r_i \, . \tag{7}$$

   Hint : A bubble nucleating in an inflationary spacetime has zero mass initially.
3. At this time, the bubble wall moves with Lorentz factor $\gamma_i = \gamma(\eta_i)$. Estimate γ_i and show that the wall motion is highly relativistic if $\rho_i \gg \sigma^2$.

   Hint : You may use the fact that we can approximate the speed of the wall as $\dot{r} \sim \gamma H r$.
4. Shortly after, the expansion of the bubble wall slows down until $\gamma = 1$ and is then surrounded by a completely empty thin layer. Explain how this can lead to the collapse of the bubble.
5. Assuming that the time needed to reach $\gamma = 1$ is negligible, show that the mass of the formed Black Hole is

   $$M_{bh} \sim \left(\frac{4}{3}\pi\rho_b + 4\pi\sigma H_i\right) r_i^3 \, . \tag{8}$$