Ludwig-Maximilians-Universität München

Solutions to

TMP-TC2 : COSMOLOGY

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Guidelines :

- The exam consists of 7 problems. You must solve problems 1 - 4.
  You must choose only one of the remaining problems 5 - 7 to solve.
- You have to upload your solutions to sync+share by Friday 28 July 20.00 CEST, at the latest.
- Please write your name AND matriculation number on every sheet that you hand in.
- You can look up in your favorite resource the values of the numerical parameters that you need.
- Your answers should be comprehensible and readable.
- It goes without saying that we expect you to work independently.

GOOD LUCK !

Problem 1	40 P
Problem 2	20 P
Problem 3	20 P
Problem 4	10 P
Extra Problem	30 P

Total	120 P

Problem 1 (40 points)

Each answer that makes sense gets 4 pt.

Problem 2 (20 points)

1. [10pt.] The equation of motion for the field follows from varying the action w.r.t. ϕ. This yields

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right] - V'(\phi) = 0\ . \tag{1}$$

Using the explicit form of the metric and taking into account that the field is homogeneous, we find

$$\ddot{\phi} + 3H\phi + V'(\phi) = 0\ , \tag{2}$$

with $H = \dot{a}/a$, the Hubble parameter.

2. [10pt.] The first Friedmann equation corresponds to the 0-0 component of the Einstein equation ; it reads

$$3H^2 = 8\pi G\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) - \frac{k}{a^2}\ . \tag{3}$$

From the above we immediately identify the energy density

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)\ . \tag{4}$$

The second Friedmann equation follows from the other components of the Einstein equation ; it reads

$$\dot{H} + H^2 = -\frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right) \tag{5}$$

Comparing the above with

$$\dot{H} + H^2 = -\frac{4\pi G}{3}\rho - 4\pi G p\ , \tag{6}$$

and using the expression for ρ, we find

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi)\ . \tag{7}$$

Problem 3 (20 points)

1. [4pt.] At $T_d = 50$ MeV, the effective relativistic degrees of freedom are the photons, the electron/positron and the neutrinos, so

$$g_\star = 2 + \frac{7}{8}(4+6) = 10.75 \ . \tag{8}$$

2. [16pt.] Decoupling happens when

$$\Gamma \approx H \ , \tag{9}$$

where the rate of the reaction is

$$\Gamma = \sigma n v \ , \tag{10}$$

with

$$\sigma = \text{cross section} \ , \tag{11}$$
$$n = \text{number density} \ , \tag{12}$$
$$v = \text{velocity} \ . \tag{13}$$

Therefore, from (9), we find

$$\sigma = \frac{H}{nv} \ . \tag{14}$$

Using

$$H = \sqrt{g_\star}\frac{T^2}{M_{\rm Pl}} \ , \tag{15}$$
$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T} \ , \tag{16}$$
$$v = \sqrt{\frac{3T}{m}} \ , \tag{17}$$

we find

$$\sigma \approx \frac{\sqrt{g_\star}}{mM_{\rm Pl}} e^{m/T} \sim 10^{67} GeV^{-2} \ . \tag{18}$$

Note that $g = 2$.

Problem 4 (10 points)

Frequency redshifts

$$\omega_{\rm em} = \frac{T_{\rm em}}{T_0}\omega_0 \ , \tag{19}$$

meaning that

$$\omega_{\rm em} \approx 10^{30}\omega_0 \ . \tag{20}$$

Problem 5 (30 points)

1. [10pt.] The particle density scales with R^{-3}. Furthermore, the scale factor is proportional to one over the temperature of a radiation-dominated universe. Therefore we can find the relation

$$n_M^0 = n_M \left(\frac{R_i}{R_0}\right)^3 = n_M \left(\frac{T_0}{T_m}\right)^3 \tag{21}$$

We can estimate the volume of a Horizon patch by $r_H^3 = \left(\frac{M_{pl}}{T_m^2}\right)^3$. So the density of the monopoles present today is given by

$$n_M^0 = \frac{1}{r_H^3}\left(\frac{T_0}{T_m}\right)^3 = \left(\frac{T_m T_0}{M_{pl}}\right)^3 \tag{22}$$

Multiplying this with the mass of the monopole gives the energy density

$$\varepsilon = m_M \left(\frac{T_m T_0}{M_{pl}}\right)^3 \sim 10^{-32} GeV^4 \tag{23}$$

Here we used $T_0 \sim 10^{-13} GeV$ and $M_{pl} \sim 10^{19} GeV$.

2. [20pt.] This part works in the same way as part two, with the difference that we have different epochs. The history is the following

$$n_M \xrightarrow{\text{radiation d.}} n_M^i \xrightarrow{\text{exponential exp.}} n_M^r \xrightarrow{\text{radiation d.}} n_M^e \xrightarrow{\text{matter d.}} n_M^0$$

Note that for radiation domination we have $R \propto T^{-1}$, for matter domination we have $R \propto T^{-\frac{4}{3}}$ and for the exponential expansion we have $R \propto e^{Ht}$. Hence, we can find

$$n_M^i = n_M \left(\frac{T_i}{T_m}\right)^3 \tag{24}$$

$$n_M^r = n_M^i e^{-3Ht} \tag{25}$$

$$n_M^e = n_M^r \left(\frac{T_e}{T_r}\right)^3 \tag{26}$$

$$n_M^0 = n_M^e \left(\frac{T_0}{T_e}\right)^{\frac{4}{3}} \tag{27}$$

We want to calculate what time for the exponential expansion is at least necessary to have the energy density of the magnetic monopoles to be equal to the critical energy density $\rho_c \sim 10^{47} GeV^4$. So we set $\rho_M = m_M n_M^0 = \rho_c$. Furthermore, we assume again that one monopole was created per Hubble patch, i.e. $n_M = \left(\frac{T_m^2}{M_{pl}}\right)^3$. Putting all these equations together and solving for the time t gives

$$t = -\frac{1}{3H} \ln\left(\frac{\rho_c}{m_M}\left(\frac{T_m^2}{M_{pl}}\right)^{-3}\left(\frac{T_i}{T_m}\right)^{-3}\left(\frac{T_e}{T_r}\right)^{-3}\left(\frac{T_0}{T_e}\right)^{-\frac{4}{3}}\right) \tag{28}$$

Inserting numbers gives

$$t \approx 1.5 \cdot 10^{-24} s \tag{29}$$

Problem 6 (30 points)

1. [5pt.] The metric is

$$g_{\mu\nu} = (\tau/\tau_0)^2 \,\mathrm{diag}(-1, 1, 1, 1) \;, \tag{30}$$

and its inverse

$$g^{\mu\nu} = (\tau/\tau_0)^{-2} \,\mathrm{diag}(-1, 1, 1, 1) \;. \tag{31}$$

2. [5pt.] The Christoffel symbols are

$$\Gamma^x_{x\tau} = \Gamma^y_{y\tau} = \Gamma^z_{z\tau} = \Gamma^\tau_{xx} = \Gamma^\tau_{yy} = \Gamma^\tau_{zz} = \Gamma^\tau_{\tau\tau} = \frac{1}{\tau} \;. \tag{32}$$

3. [5pt.] The components of the Riemann tensor are

$$R^x{}_{yxy} = R^x{}_{zxz} = R^x{}_{\tau x \tau} = R^y{}_{zyz} = R^y{}_{\tau y \tau} = R^z{}_{\tau z \tau} = \frac{1}{\tau^2} \; . \tag{33}$$

4. [5pt.] The components of the Ricci tensor are

$$R_{\tau\tau} = \frac{3}{\tau^2} \; , \quad R_{xx} = R_{yy} = R_{zz} = \frac{1}{\tau^2} \; . \tag{34}$$

   The Ricci scalar is

$$R = 0 \; . \tag{35}$$

5. [5pt.] The spacetime is not flat, since the Riemann tensor is not zero.
6. [5pt.] Using the hint, we bring the metric into the usual form in terms of a new variable t — we find this spacetime is radiation dominated since the scale factor is proportional to $t^{1/2}$.

Problem 7 (30 points)

1. [6pt.] $\rho_i = \rho(\eta_i) \gg \rho_b(\eta_i)$
2. [6pt.] As the hint says $M(\eta_i) = 0$

   In the limit where $\rho_b \ll \rho_i$, we have

$$\frac{4}{3}\pi(-\rho_i)r_i^3 + 4\pi\sigma R_i^2 \dot{r}_i \sim 0$$

$$\longrightarrow \dot{r}_i \sim \frac{1}{3}\frac{\rho_i}{\sigma} r_i$$

   which we can rewrite in terms of the Hubble parameter H to get

$$\dot{r}_i \sim \frac{1}{8\pi G\sigma} H_i^2 r_i \, . \tag{36}$$

3. [6pt.] $\dot{r} \sim \gamma H r$ implies that at time η_i

$$\dot{r}_i \sim \gamma_i H_i r_i \sim \frac{H_i^2}{8\pi G\sigma} r_i$$

$$\longrightarrow \gamma_i \sim \frac{H_i}{8\pi G\sigma}$$

   Given the assumption that $\rho_i \gg \sigma$, this is much larger than one, meaning that the bubble wall is moving at ultrarelativistic speeds.

4. [6pt.] Initially, when the bubble nucleates, it is surrounded by an FLRW universe with $\rho_i \gg \rho_b$. At that point in time, the bubble expands with its wall moving with relativistic speed.

   However, the wall slows down until it is surrounded by a thin layer without any matter in it, that is, a thin layer of $\rho = 0$. This leads to both contributions becoming positive $\longrightarrow M(\eta) > 0$. At this point, the bubble wall halts wrt the Hubble flow, which continues to expand with $H^2 = \frac{8\pi G\rho}{3}$.

5. [6pt.] If bubble is surrounded by a thin layer empty of any matter, we have $\rho = 0$ and so

$$M(\eta) \sim \left(\frac{4}{3}\pi\rho_b + 4\pi\sigma H\right) r^3$$

   Assuming that the time needed to reach $\gamma = 1$ is negligible, we can also assume that the change in r and H is relatively negligible and so the mass of the final BH is given by

$$M_{bh} \sim \left(\frac{4}{3}\pi\rho_b + 4\pi\sigma H_i\right) r_i^3 \,. \tag{37}$$