Standard Model and QCD

Problem Sheet 11

9 July 2024

Large-N limit and dimensional transmutation

In this exercise, we will study some aspects of the $O(N)$ non-linear sigma model in 2 spacetime dimensions. This toy model is asymptotically free and has a mass gap. Importantly, it can be solved at the large-N limit.
This is a theory of N scalar fields σ^a $(a = 1, \ldots, N)$ constrained by the relation

$$\sigma^a \sigma^a = 1 \ . \tag{1}$$

The action capturing the dynamics reads

$$S = \frac{1}{2g^2} \int \mathrm{d}^2 x \, \partial_\mu \sigma_a \partial^\mu \sigma_a = \frac{N}{2t} \int \mathrm{d}^2 x \, \partial_\mu \sigma_a \partial^\mu \sigma_a \ , \tag{2}$$

where $t = g^2 N$ is the so-called 't-Hooft parameter. We will be interested in the behavior of the system in the regime where N is large and at the same time g^2 is small, such that t remains fixed.

1. What is the geometrical interpretation of the constraint (2) ?
2. How many independent degrees of freedom does the theory contain ?
3. Rescale appropriately the fields σ^a in order to canonically normalize their kinetic terms. Then, build the constrained action S_λ, by inserting (1) in the action (2) through a Lagrange multiplier λ.
4. Compute the generating functional for the canonically normalized fields and integrate them out. By doing so, you get the effective action $S_{\text{eff}}(\lambda)$ for the Lagrange multiplier.
5. Compute the effective equation of motion for λ and evaluate it in momentum space, assuming that the solution is of the form $\lambda = m^2$. In the process, you will deal with a UV-divergent integral, which you can regularize with a cut-off Λ (take $\Lambda \gg m$).
6. Define the renormalized coupling

$$\frac{1}{t_{ren}} = \frac{1}{t} + 4\pi \log\left(\frac{\mu^2}{\Lambda^2}\right) \tag{3}$$

   and obtain λ as a function of the renormalized 't-Hooft coupling.
   Moreover, plug the expectation value of λ back in S_λ and convince yourselves that it actually is a mass term for the fields.
   This phenomenon goes under the name of *dimensional transmutation*, since we dynamically obtained a dimensionful parameter from a dimensionless one.