Chiral symmetry breaking and sigma model

QCD & massless quarks (up & down):

$$\mathcal{L}_{QCD} = -\frac{1}{4} G_{\mu\nu}^a G^{\mu\nu,a} + \bar{u} i \not{D} u + \bar{d} i \not{D} d$$

$$\text{w/ } D_\mu = \partial_\mu - i g A_\mu^a T^a$$

a) Define: $Q = \binom{u}{d} \rightsquigarrow Q = Q_L + Q_R = \binom{u}{d}_L + \binom{u}{d}_R$

$$\Longrightarrow \mathcal{L}_{QCD} = -\frac{1}{4} G_{\mu\nu}^a G^{\mu\nu,a} + \bar{Q}_L i \not{D} Q_L + \bar{Q}_R i \not{D} Q_R$$

In this form we can easily see that $\mathcal{L}_{QCD}$ is invariant under

$$Q_L \to L Q_L \qquad \text{w/ } \underbrace{L, R}_{\text{independent!}} \in U(2)$$
$$Q_R \to R Q_R$$

$\Longrightarrow$ global symmetry:

$$\boxed{U(2)_L \times U(2)_R \simeq SU(2)_L \times SU(2)_R \times U(1)_L \times U(1)_R}$$

↳ The Abelian part can be rewritten:

$$\begin{matrix} Q_L \to e^{i\alpha} Q_L \\ Q_R \to e^{i\beta} Q_R \end{matrix} \quad \xleftrightarrow[\beta \to \alpha - \beta]{\alpha \to \alpha + \beta} \quad \begin{matrix} Q_L \to e^{i\alpha} Q_L \\ Q_R \to e^{i\alpha} Q_R \end{matrix} \;,\quad \begin{matrix} Q_L \to e^{i\beta} Q_L \\ Q_R \to e^{-i\beta} Q_R \end{matrix}$$

or / or

$$\underbrace{Q \to e^{i\alpha} Q}_{U(1)_V} \qquad \underbrace{Q \to e^{i\beta\gamma_5} Q}_{U(1)_A}$$

The $U(1)_V$ subgroup leads to the conservation of baryon number, which (as we know) is a good quantum number in QCD.
How about the $U(1)_A$ subgroup? We do not know a further conserved charge in QCD, but $\mathcal{L}_{QCD}$ tells us that it is there. This is the starting point of the so called "$U(1)_A$ problem" (Weinberg '75) whose solution lies in the fact that $U(1)_A$ is anomalous (= broken in the quantum theory due to the non-invariance of the fermion measure in the generating functional).
$\Longrightarrow$ Only $U(1)_V$ is a good symmetry of QCD!

For N massless quarks this generalizes to

$$U(N)_L \times U(N_R) = SU(N)_L \times SU(N)_R \times U(1)_V \times U(1)_A$$

b) Let's look at the non-Abelian part. Experimentally we know that low-~~en~~energy QCD is not chiral (= left & right fermions are treated equally), but $U(2)_L \times U(2)_R$ is obviously chiral.

$\Longrightarrow$ Symmetry must be broken

$$\boxed{SU(2)_L \times SU(2)_R \longrightarrow SU(2)_V}$$

w/ $Q_L \to U Q_L$, $U \in SU(2)_V$
$Q_R \to U Q_R$

$$\boxed{\text{3 generators} + \text{3 generators} \to \text{3 generators}}$$

$\Longrightarrow$ 3 Goldstones (pions)

Similarly, $\boxed{SU(3)_L \times SU(3)_R \longrightarrow SU(3)_V}$

8 generators 8 generators 8 generators

$\Longrightarrow$ 8 Goldstones (pions, kaons, eta: Eightfold way)
Gell-Mann '61

↳ <u>Big question</u>: How is symmetry broken? What is our order parameter?

Criteria: Lorentz-scalar, $SU(3)_c$ singlet

Since there is no fundamental scalar in the spectrum we must use composite fields. Simplest possibility:

$$\boxed{\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = v^3 \quad \text{or} \quad \langle \bar{Q}^i Q^j \rangle = v^3 \delta^{ij}}$$

Check: $\langle \bar{Q}^i Q^j \rangle = v^3$ is invariant under $SU(2)_V$ but not under $SU(2)_L \times SU(2)_R$. Such an order parameter is called <u>chiral condensate</u>.

c) Goal: Find low-energy EFT. We know that $\bar{Q}Q$ is our order parameter, so let's write a Lagrangian for $\phi^{ij} = \frac{\bar{Q}^i Q^j}{v^2}$:

$$\boxed{\mathcal{L} = \frac{1}{2}|\partial_\mu \phi|^2 + \frac{\mu^2}{2}|\phi|^2 - \frac{\lambda}{4}(|\phi|^2)^2}$$

w/ $|\phi|^2 = tr(\phi^\dagger \phi)$

$\hookrightarrow \frac{dV(\phi)}{d|\phi|} = (-\mu^2 + \lambda|\phi|^2)|\phi| \stackrel{!}{=} 0 \implies \boxed{|\phi| = \sqrt{\frac{\mu^2}{\lambda}} \equiv v}$

Rewrite: $\mathcal{L} = \frac{1}{2}|\partial_\mu \phi|^2 - \frac{\lambda}{4}(|\phi|^2 - v^2)^2$

Parametrize: $\phi^{ij} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma + i\pi^1 & -\pi^2 + i\pi^3 \\ \pi^2 + i\pi^3 & \sigma - i\pi^1 \end{pmatrix}$

$$\implies tr\,\phi^\dagger\phi = \frac{1}{2} tr \begin{pmatrix} \sigma - i\pi^1 & \pi^2 - i\pi^3 \\ -\pi^2 - i\pi^3 & \sigma + i\pi^1 \end{pmatrix}\begin{pmatrix} \sigma + i\pi^1 & -\pi^2 + i\pi^3 \\ \pi^2 + i\pi^3 & \sigma - i\pi^1 \end{pmatrix}$$

$$= \frac{1}{2} tr \begin{pmatrix} \sigma^2 + \pi^a\pi^a & 0 \\ 0 & \sigma^2 + \pi^a\pi^a \end{pmatrix}$$

$$= \sigma^2 + \pi^a\pi^a$$

Vacuum mf: $\sigma^2 + \pi^a\pi^a = v^2 \rightsquigarrow S^3$

Expand around VEV:
$$\boxed{\phi^{ij} = \frac{1}{\sqrt{2}}\begin{pmatrix} (v+\sigma) + i\pi^1 & -\pi^2 + i\pi^3 \\ \pi^2 + i\pi^3 & (v+\sigma) - i\pi^1 \end{pmatrix}}$$

$\hookrightarrow \frac{1}{2}|\partial_\mu \phi|^2 = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\pi^a\partial^\mu\pi^a$

$\hookrightarrow \frac{\lambda}{4}(|\phi|^2 - v^2)^2 = \frac{\lambda}{4}((v+\sigma)^2 + \pi^a\pi^a - v^2)^2$

$$= \frac{\lambda}{4}(\sigma^2 + 2v\sigma + \pi^a\pi^a)^2$$

$$\implies \boxed{\begin{aligned}\mathcal{L} = &\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\pi^a\partial^\mu\pi^a \\ &- \frac{\lambda}{4}(\sigma^2 + 2v\sigma + \pi^a\pi^a)^2\end{aligned}}$$

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d) Historic Remark

- The above Lagrangian is reffered to as the "linear sigma model". It was coined by Gell-Mann & Levy in their famous paper "the Axial Vector Current in β-decay" ('60). They called it like that due to the σ-fluctuation appearing after SSB.
  Important: It is renormalizeable.
- Nowadays a sigma model means any theory where fields take values on a non-trivial mf.
- However, the predicted σ-fluctuation (or rather the associated particle) was never found, so it is not a good model to describe the quark condensate. In the same paper they therefore introduced the "non-linear sigma model" to get rid of the σ-particle.
  So it is actually a misnomer, there is no σ-particle in the non-linear sigma model!

Double scaling limit: $\begin{matrix} \lambda \to \infty \\ \mu^2 \to \infty \end{matrix}$ s.t. v^2 fixed

$\Rightarrow$ σ gets infinitely heavy and thus unphysical.
Pictorially, we make the walls of the mexican hat potential infinitely steep while maintaining the minimum's location & thus the S^3 vaccum mf in the above example.

$\Rightarrow$ For $V(\phi)$ not to diverge we must have

$$\boxed{\pi^a\pi^a + \sigma^2 = -2\sigma v}$$

In other words, the potential can now be seen as the constraint imposed via the Lagrange multiplier λ.

e) <u>Solve constraint explicitly:</u>

$$\sigma^2 + 2v\sigma + \pi^2 = 0$$

$$\Longrightarrow \sigma_{1/2} = -v \pm \sqrt{v^2 - \pi^2}$$

<u>Plug into $\mathcal{L}$:</u>

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{(v+\sigma)^2}{2} \partial_\mu \frac{\pi^a}{v} \partial^\mu \frac{\pi^a}{v}$$

$$- \frac{\lambda}{4} \underbrace{(\pi^a \pi^a + \sigma^2 + 2\sigma v)^2}_{=0} + \ldots \qquad \text{higher order}$$

$$= \frac{1}{2}\left(\partial_\mu \sqrt{v^2 - \pi^2}\right)^2 + (v^2 - \pi^2)\, \partial_\mu \frac{\pi^a}{v} \partial^\mu \frac{\pi^a}{v} + \ldots$$

$$= \frac{1}{2}\left(\frac{-\pi^a \partial_\mu \pi^a}{\sqrt{v^2 - \pi^2}}\right)^2 + \partial_\mu \pi^a \partial^\mu \pi^a + \ldots$$

$$= \partial_\mu \pi^a \partial^\mu \pi^a + \frac{1}{2} \frac{(\pi^a \partial_\mu \pi^a)^2}{v^2 - \pi^2} + \ldots$$

<u>Expand:</u> $\mathcal{L} = \partial_\mu \pi^a \partial^\mu \pi^a + \frac{1}{2v^2}(\pi^a \partial_\mu \pi^a)^2 + \ldots$

$$\frac{1}{v^2 - \pi^2} = \frac{1}{v^2\left(1 - \frac{\pi^2}{v^2}\right)} \simeq \frac{1}{v^2}\left(1 + \frac{\pi^2}{v^2} + \ldots\right)$$

<u>eom:</u> $-\Box \pi^a + \cancel{\frac{1}{v^2} \pi^b \partial_\mu \pi^b \partial^\mu \pi^a}$

$$- \underbrace{\frac{1}{v^2} \partial^\mu (\pi^b \partial_\mu \pi^b \pi^a)}_{} = 0$$

$$= \frac{1}{v^2} \partial^\mu \pi^b \partial_\mu \pi^b \pi^a$$

$$+ \frac{1}{v^2} \pi^b \Box \pi^b \pi^a$$

$$+ \cancel{\frac{1}{v^2} \pi^b \partial_\mu \pi^b \partial^\mu \pi^a}$$

$$\Longrightarrow \boxed{\Box \pi^a = \frac{(\partial_\mu \pi^b)^2 \pi^a}{v^2} + \frac{\pi^b \Box \pi^b \pi^a}{v^2}}$$

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Now start at given expression:

$$\mathcal{L} \supset \frac{1}{6v^2}\left((\pi^a\partial_\mu\pi^a)^2 - \pi^a\pi^a\partial_\mu\pi^b\partial^\mu\pi^b\right) + \dots$$

typo in sheet!

$$= \frac{1}{6v^2}\Big((\pi^a\partial_\mu\pi^a)^2 + \underbrace{\pi^b\partial_\mu(\pi^a\pi^a\partial^\mu\pi^b)} + \underbrace{\partial_\mu(\pi^a\pi^a\pi^b\partial^\mu\pi^b)}_{\to 0 \text{ boundary term}}\Big) + \dots$$

$$= 2(\pi^a\partial_\mu\pi^a)^2 + \pi^a\pi^a\pi^b\Box\pi^b$$

$$\overset{\text{eom}}{=} 2(\pi^a\partial_\mu\pi^a)^2 + \underbrace{\pi^a\pi^a\pi^b\left(\frac{(\partial_\mu\pi^c)^2\pi^b}{v^2} + \frac{\pi^c\Box\pi^c\pi^b}{v^2}\right)}_{\in\mathcal{O}(\pi^6)}$$

$$= \frac{1}{6v^2}\left((\pi^a\partial_\mu\pi^a)^2 + 2(\pi^a\partial_\mu\pi^a)^2\right) + \dots$$

$$= \frac{1}{2v^2}(\pi^a\partial_\mu\pi^a)^2 + \dots$$

Chiral Lagrangian

f) Exponential representation:

$$U = \exp\left(i\frac{\pi^a\sigma^a}{f_\pi}\right) \Longrightarrow \boxed{\mathcal{L}_{\text{chiral}} = \frac{v^2}{4}\mathrm{Tr}(\partial_\mu U^\dagger\partial^\mu U)}$$

Intuition: all these "representations" are nothing else than different coordinates on the S^3. Previous rep were stereographic coords!

Obviously invariant under global trafo:

$U \to L U R^\dagger$ where $L, R \in SU(2)$ independent!

After SSB we are left with $L = R \in SU(2)_V$ and as we showed by expanding the three exponentials in some previous sheet, the Goldstones transform in the adjoint rep of the algebra

$$\boxed{\delta^a\pi^c = i\frac{\pi^b}{v}f^{abc}}$$

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Note that by expanding (8) using (7) one finds (6). However, you will need the second order terms of the exponentials to get the prefactors right.

g) • $U^\dagger U = 1$

• U-tadpole, i.e. something like ΘU can be eliminated by redefining U. In other words, U-term leads to a VEV which can be taken care of by expanding around right minimum.

$\Longrightarrow$ Only other building block for $\mathcal{L}_{chiral}$ are derivatives. Thus we can organise $\mathcal{L}_{chiral}$ by $\#\partial$.

h) Add $\delta\mathcal{L}_{mass} = v^3\, tr(MU + M^\dagger U^\dagger)$ as perturbation to "slightly" break the $SU(2)_L \times SU(2)_R$ symmetry explicitly. This gives the Goldstones a small mass (in this case we call them pseudo-Goldstones).

↳ For formal invariance of $\mathcal{L}$, we must have

$$\boxed{\begin{aligned} M &\to R^\dagger M L \quad \text{as} \\ U &\to L U R^\dagger \end{aligned}}$$

↳ In general the mass is a complex number in the Lagrangian. However, we can make it real by a chiral transformation.

Consider e.g. the Dirac-Lagrangian

$$\mathcal{L}_D = \bar\Psi_L i\not\partial \Psi_L + \bar\Psi_R i\not\partial \Psi_R - \mu \bar\Psi_L \Psi_R - \mu^* \bar\Psi_R \Psi_L$$

Let's write $\mu = m e^{i\theta}$ and perform the chiral rotation

$$\Psi_L \to e^{i\frac{\theta}{2}} \Psi_L$$
$$\Psi_R \to e^{-i\frac{\theta}{2}} \Psi_R$$

This removes the complex phase. This is however not possible in the presence of a non-Abelian gauge field, since the non-invariance of the fermion measure induces a term $\delta\mathcal{L}_\theta' = \theta \frac{g}{32\pi^2} G_{\mu\nu}^a \tilde{G}^{\mu\nu,a}$, which due to the presence of Yang-Mills instantons is physical and contributes e.g. in the electric dipole moment of the neutron. In other words, the necessary $U(1)_A$ symmetry is anomalous.

$\Longrightarrow$ "Strong CP problem"

↳ Let's set $\theta = 0$ (for our purpose reasonable, experimentally $|\theta| < 10^{-9}$) and expand U in inverse powers of f_π:

$$U = \exp\left(\frac{i\pi^a\sigma^a}{f_\pi}\right) = 1 + \frac{i\pi^a\sigma^a}{f_\pi} - \frac{\pi^a\pi^b}{2f_\pi^2}\sigma^a\sigma^b + \dots$$

$$\Longrightarrow v^3 \, tr\left[M(U + U^\dagger)\right]$$

$$= v^3 \, tr\left[M\left(1 + \cancel{\frac{i\pi^a\sigma^a}{f_\pi}} - \frac{\pi^a\pi^b}{2f_\pi^2}\sigma^a\sigma^b + \dots + 1 - \cancel{\frac{i\pi^a\sigma^a}{f_\pi}} - \frac{\pi^a\pi^b}{2f_\pi^2}\sigma^a\sigma^b + \dots\right)\right]$$

$$= -v^3 \, tr\left[M \frac{\pi^a\pi^b}{f_\pi^2} \underbrace{\sigma^a\sigma^b}_{= \delta^{ab}\mathbb{1} + \text{antisym part, vanishes when contracted with } \pi^a\pi^b} + \dots\right]$$

$$= -\frac{v^3}{f_\pi^2} \, tr(M) \, \pi^a\pi^a$$

$$\Longrightarrow \boxed{m^2 = \frac{2v^3(m_u + m_d)}{f_\pi^2}}$$

Gell-Mann - Oakes - Renner relation

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